Limit Topology in Vertex Operator Algebras of Higher Genus

(Based on Discussion with Nicolai Reshetikhin)

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Abstract

We prove: Polynomial-time solvable, vertex correlation Pf$((\Gamma^K)^{-1}) \in \textbf{Quot}(\mathbb{K}[D])$ for all genus $g \gg$, fluctuating dual spanning tree projective-height, bipartite partition $\overline{\mathcal{M}}_g \supset (\mathbb{Z}^+)^k$; and, the free Dirac Fermion convergence $\Psi = f \cdot (1 + O(1))$ in thermodynamic scaling limit of Grassmann-kernel asymptotics-discriminant steepest descent. We get conjecture for the large deviation functional, Green's function $\langle \cdots \rangle$ of Dirichlet problem by steepest decent and variational principle.

Keywords: Limit-topology, Vertex-operator-algebras, Higher-genus

1. Characterizations

(i) Derive bipartite partition function Z using Grassmann integral

(ii) Prove polynomial-time, vertex correlation $\forall\, g$-bipartite partition

2. Special cases

(i) Reformulate the Grassmann integral for special genus-g domains

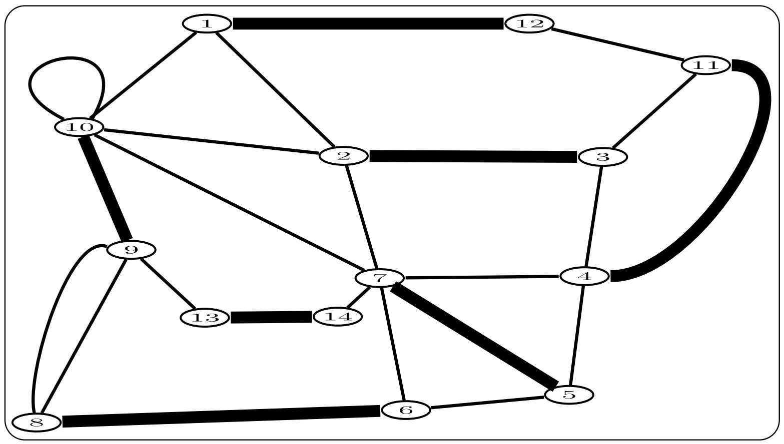
(ii) Find thermodynamic $\ln(\cdot)$ steepest-descent and variational-principle

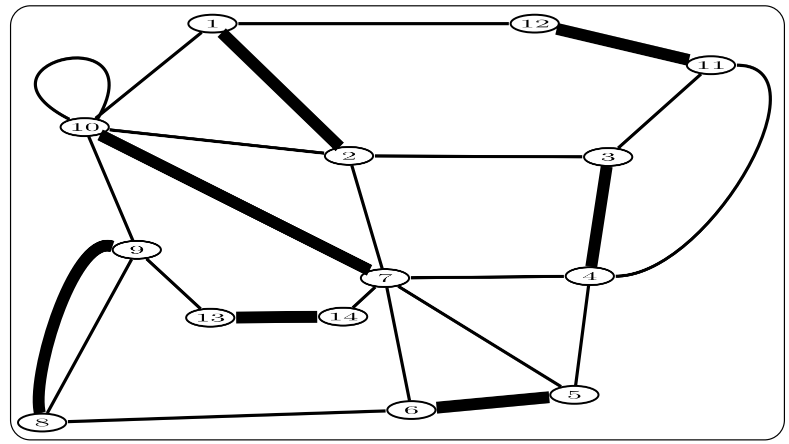
(iii) State conjecture for large deviation functional, Green's function $\langle \cdots \rangle$

1 Characterizations

1.1 Basic definitions and observations

$\forall\, g \gg,$ an embedded surface graph $\Gamma = (i_\ell \,|\, \ell' \geqslant \ell \in \mathbb{N}^+,\, i_{\ell'} \neq i_\ell) \subset \overline{\mathcal{M}}_g$.



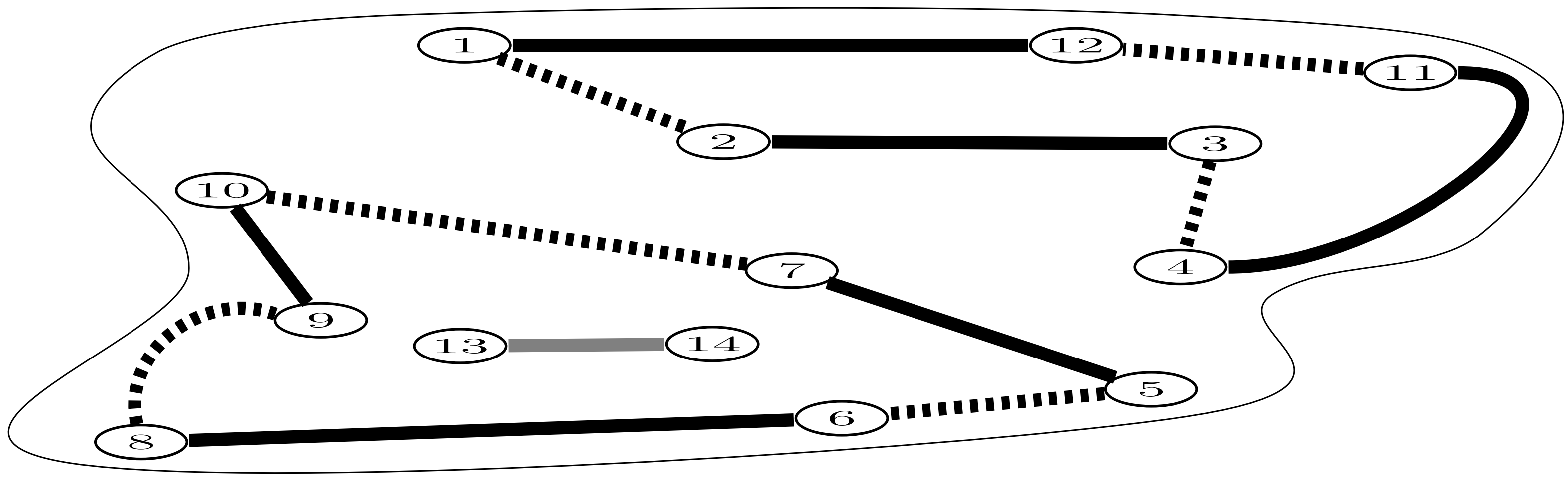


$\Gamma \cong$ partition $\boldsymbol{\sigma} \in \mathrm{Aut}(\mathscr{D}) \iff$ perfect-matching $D \subseteq \mathscr{D} = (D,\, \forall\, \ell \,|\, i_\ell \neq j_\ell)$
i.e. ✓ Dimers $(\ell)_{\ell \in D}$ do not overlap, and ✓ All vertices are covered, by:

$$\sum_{\ell \neq \ell'} \boldsymbol{\sigma}_D(i_\ell, j_\ell) \;=\; \frac{1}{2}\left|\partial D = \bigsqcup_{\ell} (i_\ell \,|\, \ell' \geqslant \ell \in \mathbb{N}^+)\right| \;\left|\; \boldsymbol{\sigma}_D(i_\ell, j_\ell) = \begin{cases} 1 \text{ if } \ell \in D \\ 0 \text{ if } \ell \notin D. \end{cases}\right.$$

Remark. $\overline{\mathcal{M}}_g =$ closed compact orientable; $\bigcap_{\ell \in D} (i_\ell \,|\, i = 1, 2, \ldots) = \emptyset$; and, the embedding $=$ cell-complex, i.e. face $\approx$ topological disk, i.e. no hole.

The transition subgraph = symmetric-difference $D_1 \Delta D_2 = D_1 \cup D_2 \setminus D_1 \cap D_2$



i.e., homology $\mathcal{H}^1(\Gamma;\mathbb{Z}_2) = \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$ class of 1-cycle = union of ordered even-length $\eta = \sum_{C_\alpha} \sigma_{C_\alpha}(D_1 \Delta D_2)$ transition cycles, the simple closed paths:

$$C_\alpha = (\ell_{2n_{\alpha-1}+1}, \ldots, \ell_{2n_\alpha}), \quad \forall\, \alpha \in \mathbb{N}^+ \,|\, 1 \leqslant \alpha \leqslant \eta \quad \left|\begin{array}{l} (\ell_{2n_{\alpha-1}+1}, \ell_{2n_{\alpha-1}+3}, \ldots, \ell_{2n_\alpha-1}) \in D_1 \\ (\ell_{2n_{\alpha-1}+2}, \ell_{2n_{\alpha-1}+4}, \ldots, \ell_{2n_\alpha}) \in D_2 \end{array}\right.$$

traversing $(i_{2n_{\alpha-1}+1}, \ell_{2n_{\alpha-1}+1}, \ldots, i_{2n_\alpha}, \ell_{2n_\alpha})$

and $n_0 = 0.$

Remark. D_1, D_2 are equivalent if $D_1 \Delta D_2 = 0 \in \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$; $\forall\, D \in \mathcal{C}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2) = 1$-chain in cell-complex over $\mathbb{Z}_2 \;\big|\; \partial D \in \mathcal{C}^0(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$.

Local observable = dimer-dimer correlation (conditional probability)

$$\left\langle \prod_{i=1}^{k} \sigma_D(\ell_i) \right\rangle \stackrel{\text{def}}{=\!=} \text{Prob}(\ell_1 \in D, \ldots, \ell_k \in D) = \mathbb{E}\Big[\prod_{i=1}^{k} \sigma_D(\ell_i) \Big]$$

$$= \sum_{D \subseteq \mathscr{D}} \prod_{i=1}^{k} \sigma_D(\ell_i) \times \text{Prob}(D) = \frac{\sum\limits_{D \subseteq \mathscr{D}} \prod\limits_{i=1}^{k} \sigma_D(\ell_i) \prod\limits_{\ell \in D} \varepsilon_\ell^K \omega_\ell}{\sum\limits_{D \subseteq \mathscr{D}} \prod\limits_{\ell \in D} \varepsilon_\ell^K \omega_\ell}$$

$$= \frac{1}{Z} \sum_{D \subseteq \mathscr{D} \,|\, \ell_1, \ldots, \ell_k \in D} \varepsilon_D^K \, \omega_D \; \left| \begin{array}{l} \omega_{(\cdot)} = \prod\limits_{\ell \in (\cdot)} e^{-\frac{\Xi_{(\cdot)}}{\kappa T}} \quad \text{(Boltzmann)} \\ \varepsilon_{(\cdot)}^K = \prod\limits_{\ell \in (\cdot)} \varepsilon_\ell^K = \pm 1, \quad \Xi_{(\cdot)} = \sum\limits_{\ell \in (\cdot)} \Xi_\ell \end{array} \right.$$

positive semi-definite for $Z > 0$, weight $\omega_{(\cdot)} > 0$, and for all vector

$$\mu = (\mu_h = \mathbb{E}[D_h]).$$

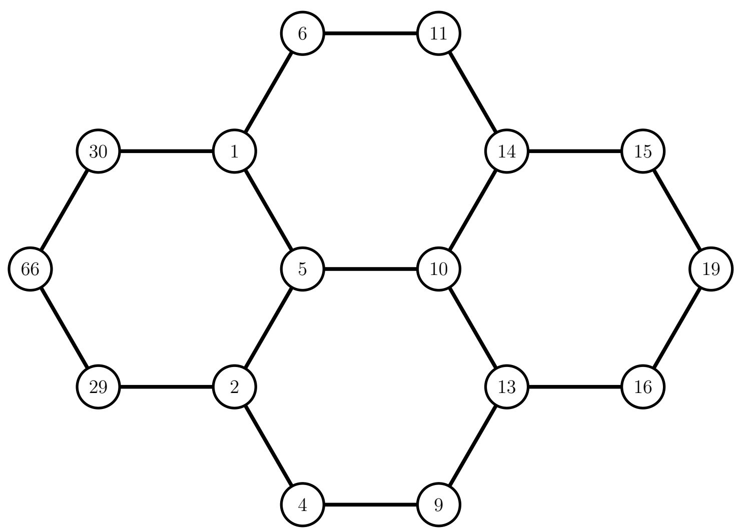
Remark. $\langle \cdots \rangle = 0 \Longleftrightarrow \text{Prob}(\cdot) = 0$ if $(\ell_\xi, \ell_\eta) \,\big|\, \xi \neq \eta$ share common vertex; moreover, $\left\langle \cdots \right\rangle \Big| \, k = n \Longrightarrow$ normalization.

That is, by $\pm$ signs and $\Xi_{(\cdot)}$, graph $\Gamma \overset{\lambda}{\sim}$ dimer σ-finite probability measure $\lambda(\cdot) \in \Xi : E(\Gamma) \longrightarrow \mathbb{R} \mid \ell \longmapsto \Xi_\ell \iff \Gamma$ is (Boltzmann) weighted:

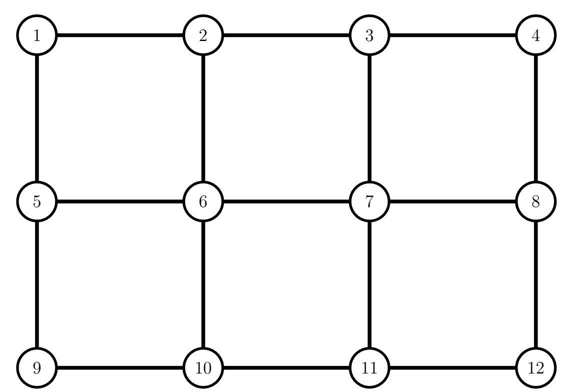
$$\left\langle \prod_{i=1}^{k} \sigma_D(\ell_i) \right\rangle = \frac{\prod\limits_{\ell \in D} \varepsilon_\ell^K \omega_\ell}{\sum\limits_{D \subseteq \mathscr{D}} \prod\limits_{\ell \in D} \varepsilon_\ell^K \omega_\ell} = \frac{\varepsilon_D^K \omega_D}{Z} = \text{Prob}(D) \Big| \omega_{(\cdot)} = e^{-\frac{\Xi_{(\cdot)}}{\kappa T}},\ \varepsilon_{(\cdot)}^K = \prod_{\ell \in (\cdot)} \varepsilon_\ell^K$$

$$= \frac{1}{Z}\, \varepsilon_D^K \exp\left(-\frac{\Xi_D}{\kappa T}\right) \Big| Z = \sum_{D \subseteq \mathscr{D}} \varepsilon_D^K \exp\left(-\frac{\Xi_D}{\kappa T}\right),\ \Xi_{(\cdot)} = \sum_{\ell \in (\cdot)} \Xi_\ell$$

where $Z =$ strict-sense positive, continuous partition function on objects:



- Domains in regular hexagonal grid.

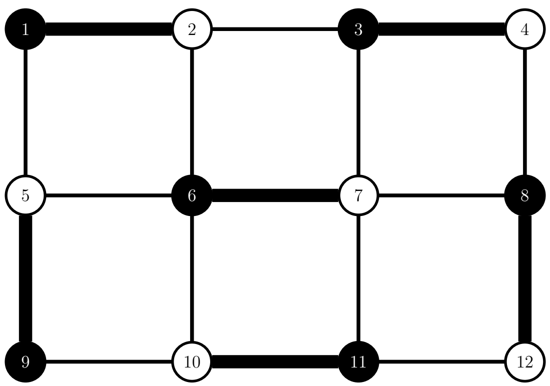


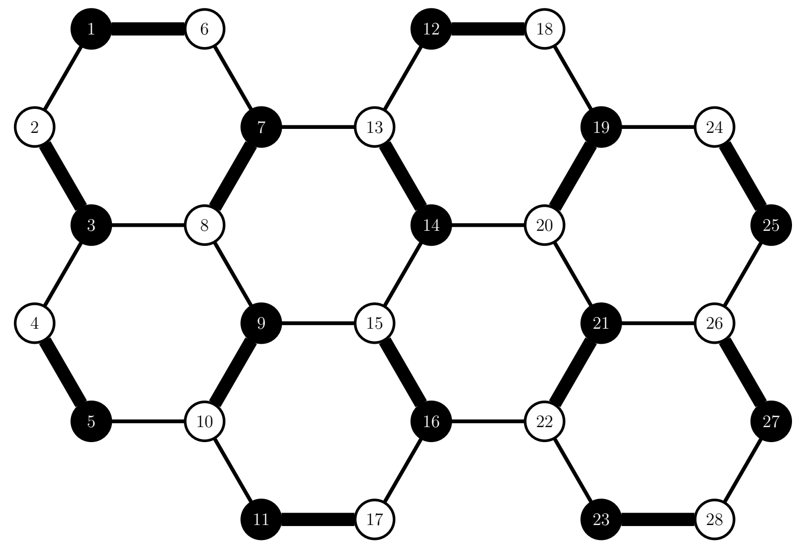
- Domains in square grid.

Remark. Bipartite $\Longrightarrow$ no adjacent-black or -white vertices, for well-defined path cartesian product: $M\times N$ vertices ($(M-1)\times(N-1)$ edges), $2n=MN$ quadratic lattice

$$V(\Gamma) \;=\; V_\bullet(\Gamma) \sqcup V_\circ(\Gamma).$$

Instance.





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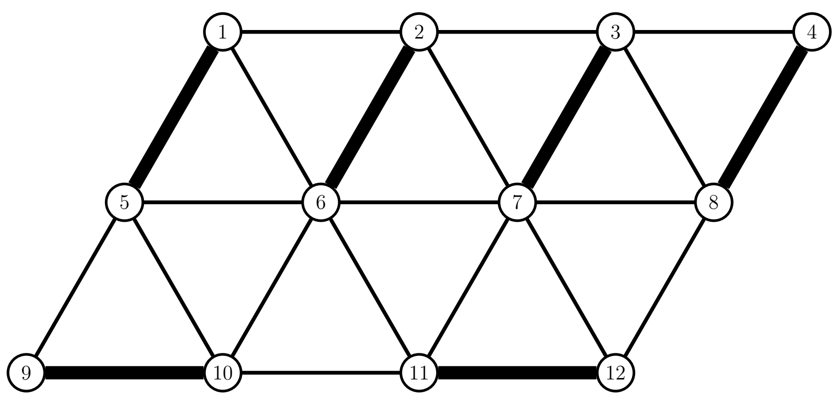
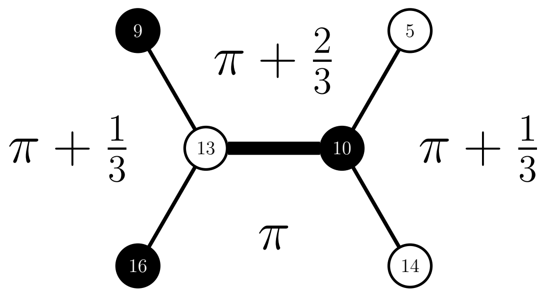
Non-instance.

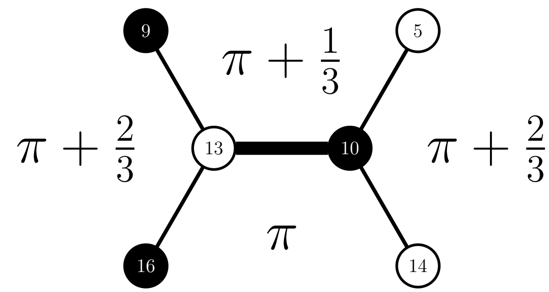


$\left(\begin{array}{l}\textit{no bipartite structure}\\ \textit{on triangular grids}\end{array}\right).$

The space of height functions is generated by orthogonal projections:

$$\mathcal{H}_\Gamma \overset{\text{def}}{=\!=} \{\pi :\ \text{faces}(\Gamma) \longrightarrow \mathbb{Z}\}$$





for all reference face f_0 boundary-normalization $\pi(f_0) = 0$;

where $\Gamma \subset \mathbb{R}^2 =$ bipartite, hexagonal embedding of

$$\boldsymbol{Dimers} \longleftarrow \boldsymbol{Discrete\ surfaces}.$$

Proposition (boundary-face).

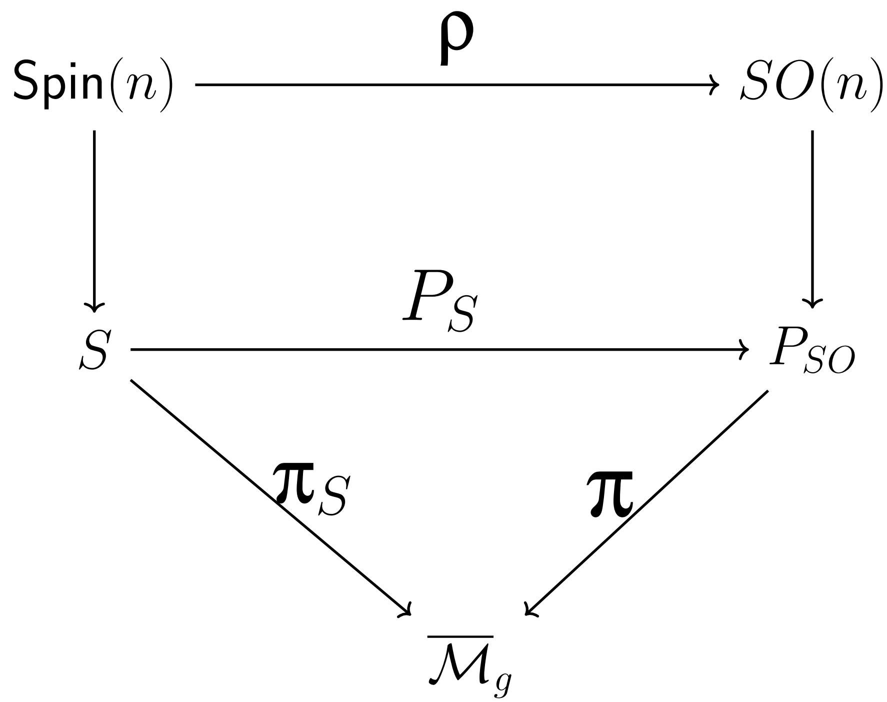
(i) $\pi_D\big|_{\partial\Gamma} = \pi_D$ *restricted to boundary-faces* $\partial\Gamma$ *is independent of* D.

(ii) $\pi_{D_1 D_2} = \pi_{D_1} - \pi_{D_2}$.

Proof. $\heartsuit$.

The spin structure = oriented orthonormal frame $P_{SO} \longrightarrow \overline{\mathcal{M}}_g$ equivariant lift of double covering $\rho\colon \text{Spin}(n) \longrightarrow SO(n) \mid \overline{\mathcal{M}}_g$ = orientable surface;

i.e., equivalent to the pair (S, P_S) equivariant 2-fold cover $P_S\colon S \longrightarrow P_{SO}$ frames $\pi_S\colon S \longrightarrow \overline{\mathcal{M}}_g$ of principal Spin(n)-bundle $\pi\colon P_{SO} \longrightarrow \overline{\mathcal{M}}_g$:



$$\pi_S = \pi \circ P_S$$

$$P_S(p\,q) = P_S(p)\ \rho(q)$$

$$p \in S, \quad q \in \text{Spin}(n).$$

Remark. Spin(n) = spinor-space Δ_n structure group; S = spinor bundle; complex vector bundle $\equiv$ spinor-bundle frames $\pi_S\colon S \longrightarrow \overline{\mathcal{M}}_g$ of principal Spin(n) bundle $\pi\colon P_{SO} \longrightarrow \overline{\mathcal{M}}_g$.

1.2 Timeline

1.2.1 Number of $\pm$ Pfaffians

Kasteleyn (1963): $\forall\, g\!=\!0,\; Z\!=\pm$ Pfaffian of Kasteleyn matrix.

Kasteleyn (1963): $\forall\, g\!=\!1,\; Z\!=$ linear in 4 Pfaffians; 3 "$+$", 1 "$-$".

Kasteleyn (1963): $\forall\, g\!>\!1,\; Z\!=$ conjecture; mysterious 2^{2g} Pfaffians; project was not finished, at least, not published.

1.2.2 Combinatorics of $\{\pm\}$

Gallucio & Loebl (1999): $Z := \pm 1$ for compact orientable surface.

Tesla (2000): $Z := (\sqrt{-1}, \pm 1)$ for non-orientable surface; number of Paffians is more or less same.

Cimazoni & R. (2004 - 05): $Z := \pm 1$ by spin-structures.

Cimasoni (2006): $Z := \sqrt{-1}$ by orientable double-cover pin-minus structure; $\cong\ \pm 1$ for spin structure; a topological model for G. Tesla (2000).

1.2.3 Pfaffian Asymptotics

R. et al. (2000s): By sum over height functions $h(\mathcal{F})$ with face-weights $q_{\mathcal{F}}$,

$$Z = \sum_{D} \prod_{\ell \in D} \omega_\ell \ = \ \text{Const.} \times \sum_{h} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h(\mathcal{F})} .$$

With entropy (as $|\Gamma| \longrightarrow \infty, \ q_{\mathcal{F}} \longrightarrow 1$) given by Gaussian field theory (Seiberg-Witten conjecture), and by path integral in the scaling limit,

$$Z = \int \exp\left\{ -\frac{1}{2}\left(\int_{\overline{\mathcal{M}}_g} (\partial\Phi)^2 \, d^2x \ + \int_{\overline{\mathcal{M}}_g} \lambda(x)\, \Phi(x) \right)\right\}$$

where R.H.S linear multiple $\lambda(x)\,\Phi(x)$ is given in terms of $q_{\mathcal{F}}^{h(\mathcal{F})}$ by:

$$q_x \ = \ \ell^{-\varepsilon\ \lambda(x)} \quad \Big| \ \varepsilon \ = \ \text{lattice step; } \ \lambda \ = \ \text{logarithmic scale, as } \varepsilon \longrightarrow 0.$$

Moreover, by Alvarez-Gaumé, Moore, Nelson, & Vafa (1986), studying Fermi and Bose partition correspondence on Riemann surfaces,

$$\text{R.H.S.} \ \sim \ \sum_{\xi \in S(\overline{\mathcal{M}}_g)} \text{Arf}(\xi) \times |\Theta(z \mid \xi)|^2 \quad \Big| \ z \ = \ \text{determined by } \omega\text{'s.}$$

Remark. Natural conjecture is born for Pfaffians of critical ω in linear decay of correlation in large thermodynamic scaling limit: the Z asymptotics

$$e^{\textsf{Volume}} \times \textbf{ the free energy}$$

where next leading term is sum of theta functions, and square of each theta function is next leading asymptotics of each Pfaffian, respectively.

It was confirmed in:

(i) *Ferdinand (1967):* For square-grid torus.

(ii) *Costa-Santos & McCoy (2002):* For genus $\geqslant 2$, numerically:

$$\mathrm{Arf}(\xi) \times |\Theta(z \mid \xi)|^2 .$$

That is, the natural conjecture works, however, still conjecture: No proof yet.

Remark. (i) Correlation equals logarithmic ω derivative, but entropy model is practical sophistication for a fixed (rather than varying) genus.

(ii) $Z =$ glueable (summable) on boundary spins for surfaces with boundary.

(iii) "Higher" spin structure is unknown, perhaps a para-polynomial theory.

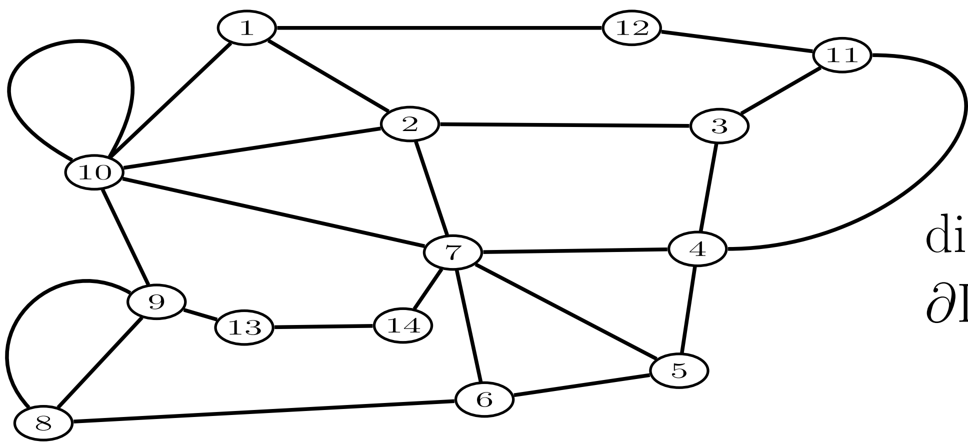
1.3 Combinatorial Equivalence

Lemma (coloring-isomorphism). *Given space of tilings, resp. dimers,*

$$\boldsymbol{family\ (Dimers)} \longleftrightarrow \boldsymbol{family\ (Tilings)}.$$

Proof. Let $\Gamma \subset \mathbb{R}^2$ = closed connected, planar (non-intersecting edges); the set of all Γ spanning trees defines:

(i) For 2D cell complex Γ:
0-cells, 1-cells, 2-cells = vertices, edges, faces, resp.

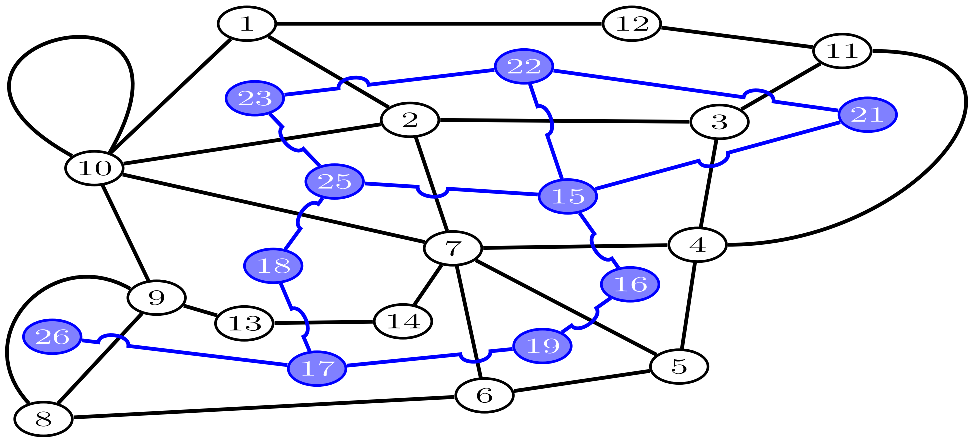


Disjoint interiors.

$\dim(\partial\Gamma^{(k)}) = (k-1) \bmod 2$
$\partial\Gamma^{(k)}$ = *boundary of two k-cells,*
$k = 0, 1, 2.$

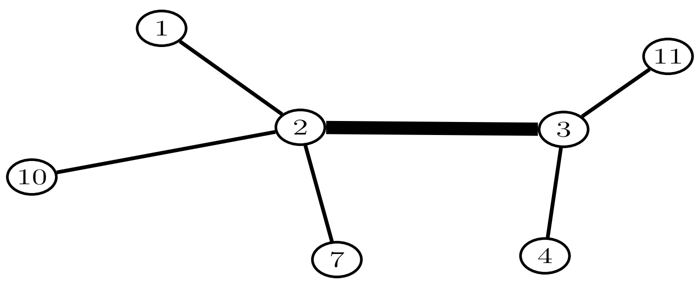
Remark. $\Gamma \subset \overline{\mathcal{M}}_g$ = embedding i.e. generally, 1-skeleton CW-complex (resp. cell-decomposition of orientable, closed connected genus g surface).

(ii) For dual cell complex Γ^* (the set of all Γ^* spanning trees):
0-cells, 1-cells, 2-cells $=$ "*centers*" of 2-cells, 1-cells, 0-cells of Γ, resp.

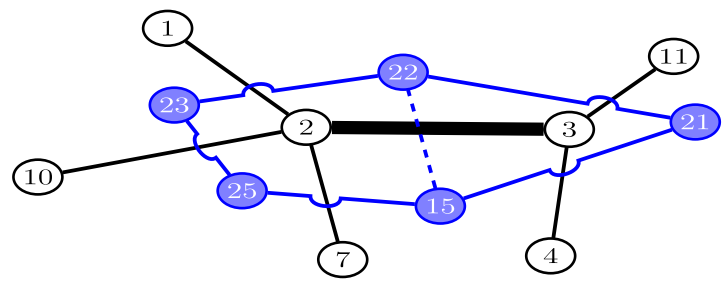


$\Gamma^* =$ dual cell complex to Γ.

(iii) For a dimer on Γ:



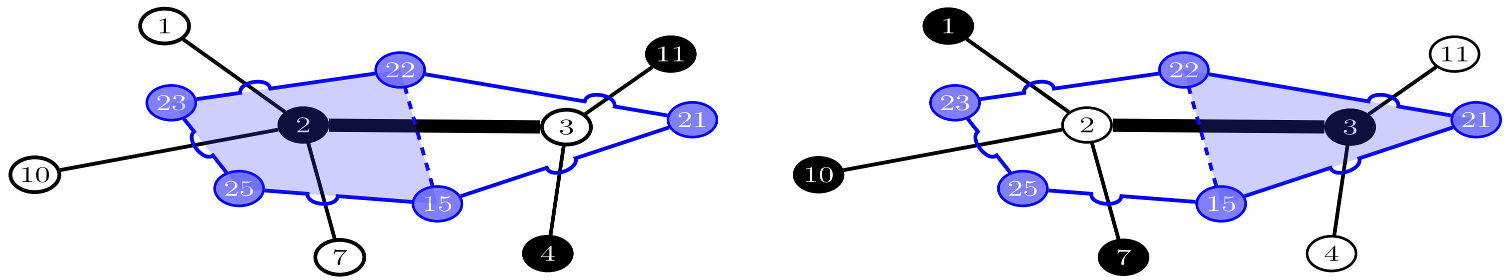
Unique pair of 2-cells on Γ^* share:



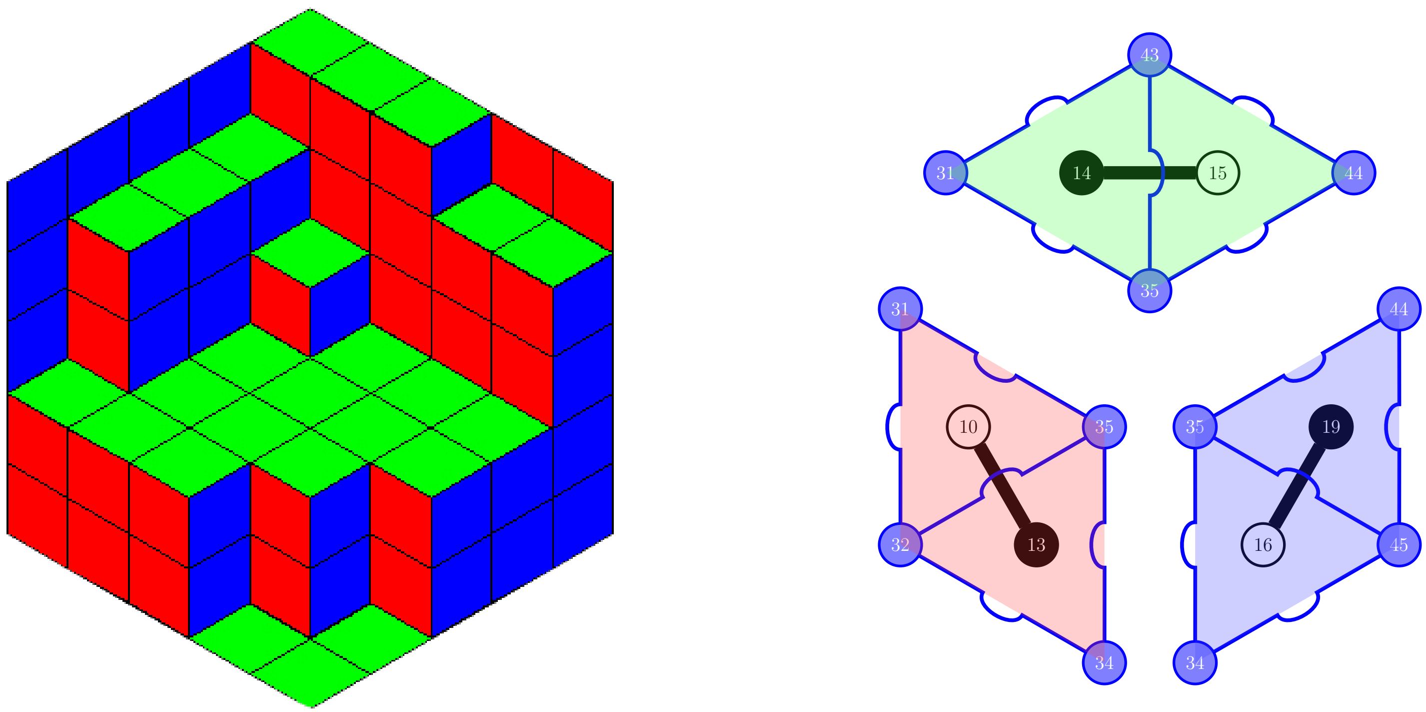
(iv) Therefore, the global bijection:

$$(\text{Dimers on } \Gamma) \quad \longleftrightarrow \quad \begin{pmatrix} \text{Tilings of } \Gamma^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells} \end{pmatrix}. \qquad \square$$

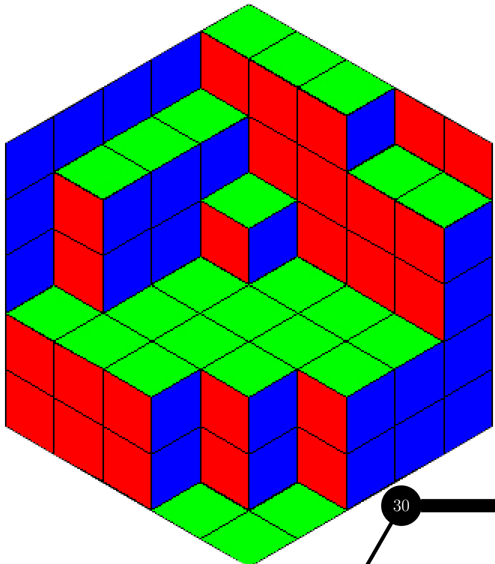
Remark. On bipartite graph:

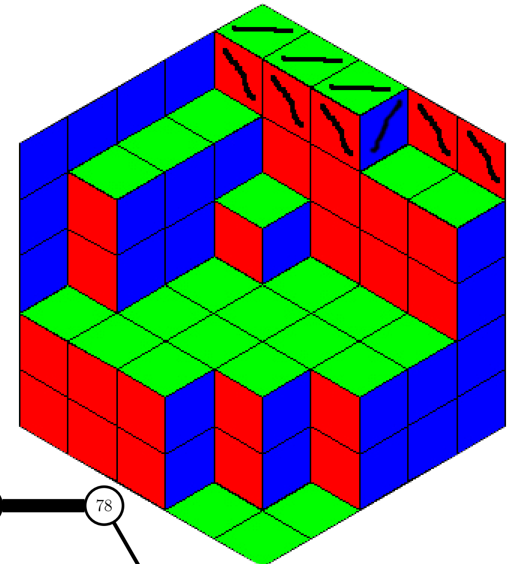


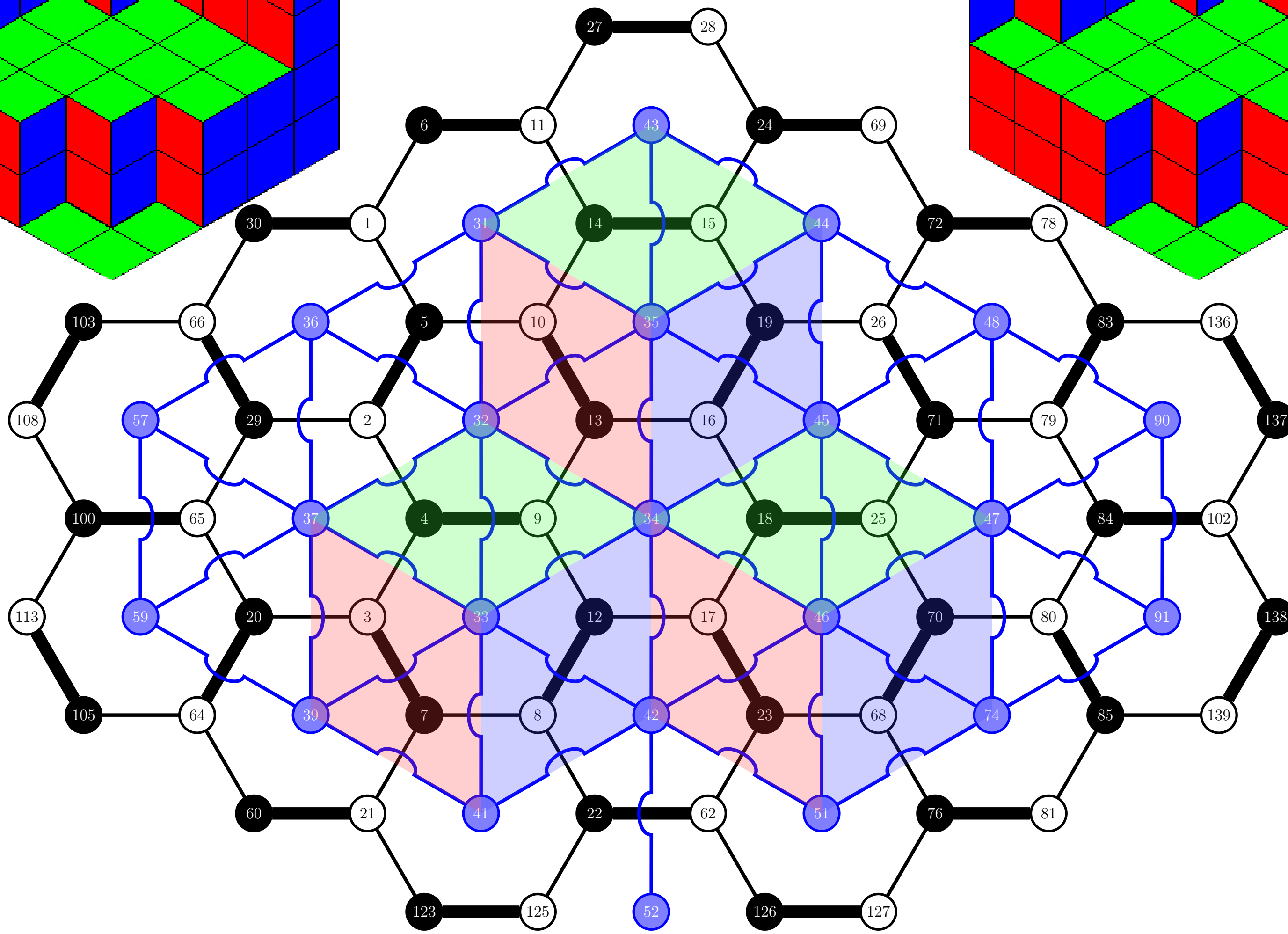
("two-color" tiles, above-shown, are admissible)



Cubes: 3D boxes by 2D rhombus-tiling projection







Theorem. *Dimer probability* Prob$(D) = (1/Z)\prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi_D(\mathcal{F})}$ *equals measure*

$$\text{Prob}(\pi) = \frac{1}{Z}\prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi(\mathcal{F})} \;\Big|\; Z = \sum_{\pi\in\mathcal{H}_\Gamma}\prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi(\mathcal{F})},\; q_{\mathcal{F}} = \prod_{\ell\in\partial\mathcal{F}} \omega_\ell^{\varepsilon_\ell^K}.$$

Proof. The combinatorial equivalence i.e. coloring-isomorphism

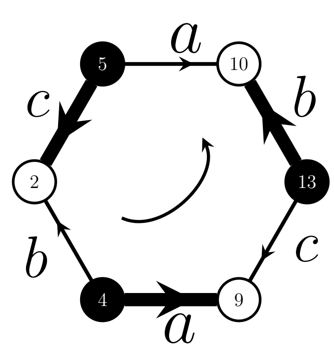
$$\Big\{\textit{Dimers on } \Gamma\Big\} \underset{\textit{bijection}}{\cong} \Big\{\textit{height functions}\Big\}$$

with the boundary-face proposition $\Longrightarrow$ Prob(D) = Prob(π). □

Remark. Prob(D) = "gauge" invariant measure: $\omega_\ell \longmapsto s(\ell_+)\;\omega_\ell\; s(\ell_-)$. Furthermore, $q_{\mathcal{F}}$ = invariant ("essential" parameters).

Particular cases.

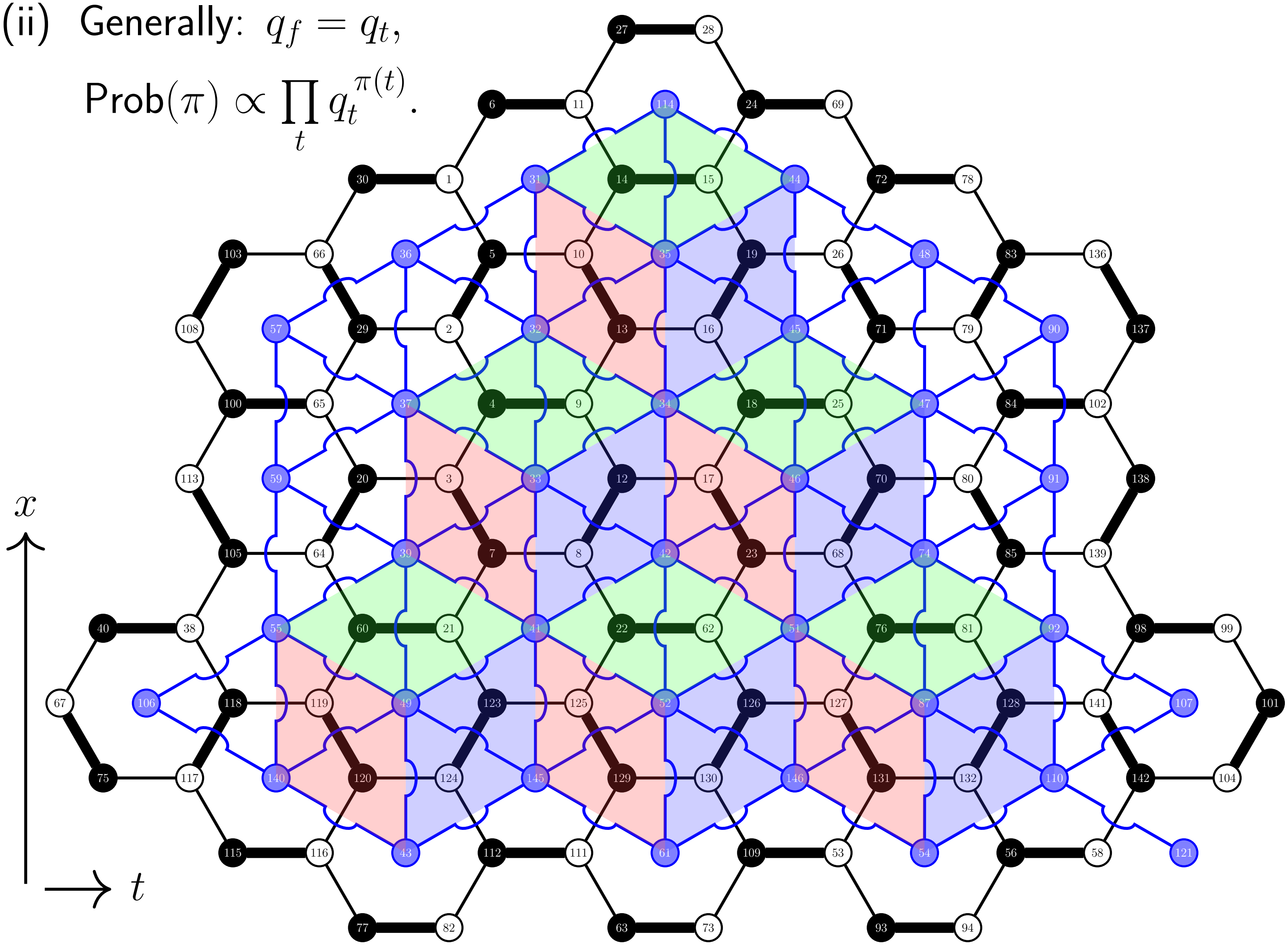
(i) Uniform distribution:



$$q \;=\; a^{-1}b\,c^{-1}a\,b^{-1}c \;=\; 1.$$

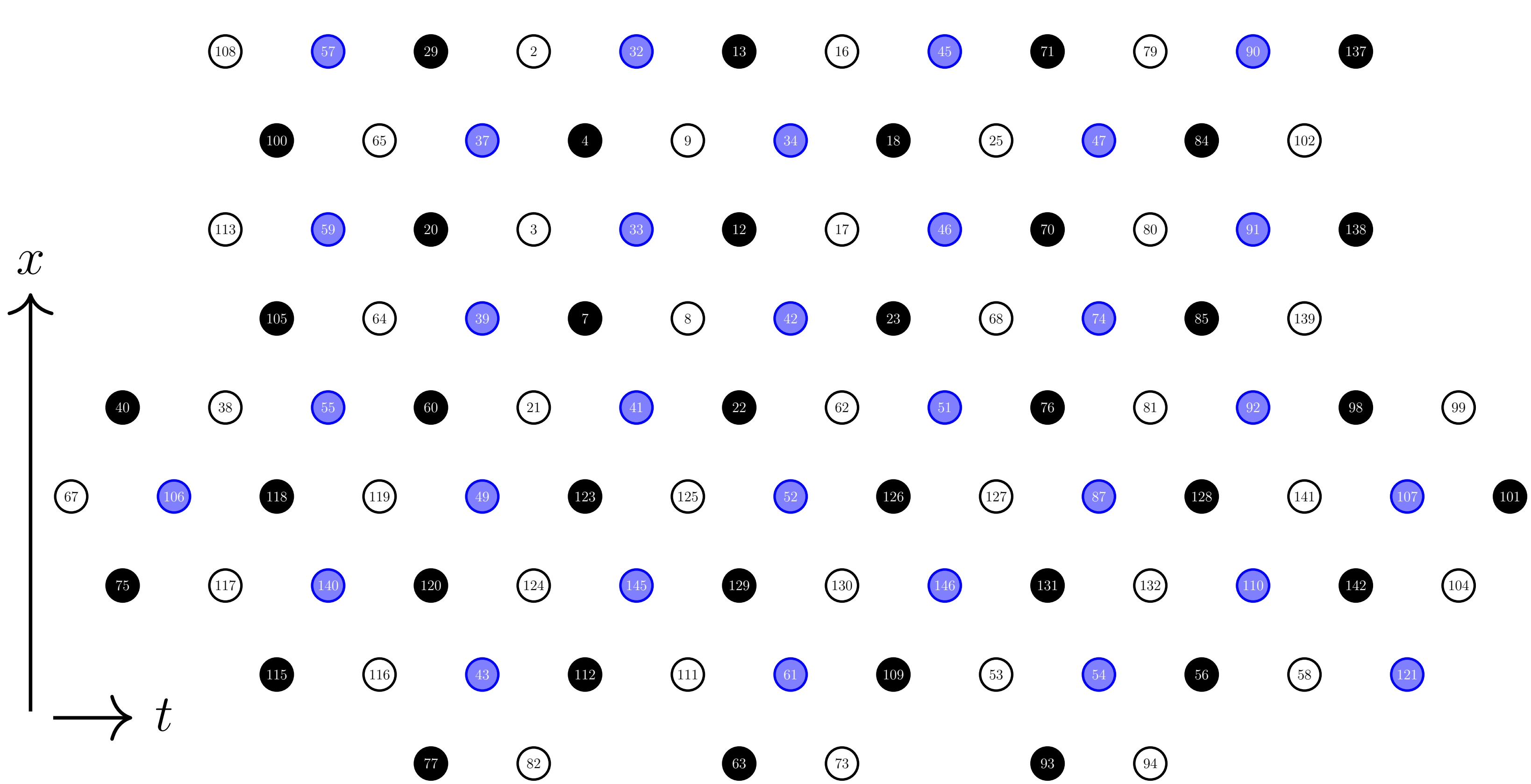
(ii) Generally: $q_f = q_t$,

$$\text{Prob}(\pi) \propto \prod_t q_t^{\pi(t)}.$$



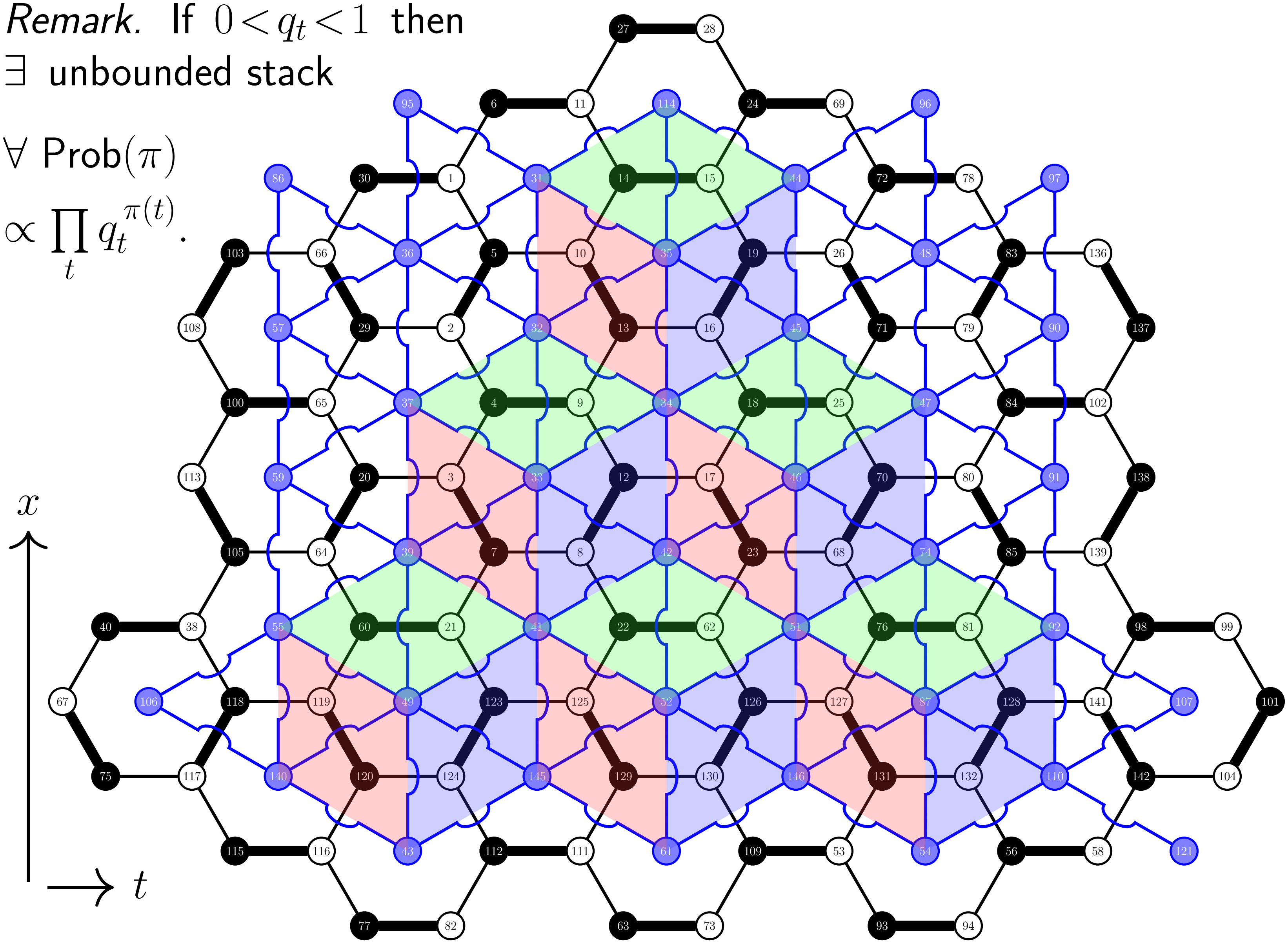
Remark. If $0 < q_t < 1$

$\forall$ Prob(π) $\propto \prod_t {q_t}^{\pi(t)}$.



Remark. If $0 < q_t < 1$ then
$\exists$ unbounded stack

$\forall$ Prob(π)
$\propto \prod_t q_t^{\pi(t)}$.



Lemma (perfect-matching). $\forall\,\mathscr{D}$, *rank* $|\{\widetilde{\sigma}\}|$ *of equivalence classes*,

(i) $|\{\widetilde{\sigma}\}| \leqslant \sqrt{(2n)! \cdot 2^{-((1/\varepsilon)\ mod\ d(n))} \cdot e^{\ln(g(n)\cdot f(n))}}$, $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$.

(ii) $\min(\deg(\Gamma)) \geqslant (2n-1)$, $\forall\,\langle g(n), f(n)\rangle \in \mathbb{R}$,

$$\iff g(n)\cdot n!\cdot e^{\ln f(n)} = (2n-1)!! = (2n-1)\cdots 3\cdot 1 = \prod_{k=0}^{n-1} 2k+1.$$

Proof. Albeit perfect-matching rank $|\mathscr{D}|$, $\mathcal{S}_{2n} \supseteq \mathrm{Aut}(\mathscr{D}) = \{\widetilde{\sigma}\}\times(\mathcal{S}_n \times \mathcal{S}_2^n)$:

$$\sum_{D(N_1,\dots,N_k)\ |\ (\sum_{\nu=1}^{k} N_\nu)=n} 1 \ \Bigg|\ \widetilde{\sigma}\colon \Gamma \longrightarrow \Gamma \begin{cases} \sigma(1,\dots,2n) = (\sigma(1),\dots,\sigma(2n)) \\ \sigma(2\ell) > \sigma(2\ell-1),\ \forall\ \ell \in \mathbb{N}^+ \end{cases}$$

where $d := N_\nu = |\text{edges}|$, $\forall\,\nu$-class; $|\{\widetilde{\sigma}\}| = \Big|\{[\sigma]\} \subseteq \mathcal{S}_{2n}\big/(\mathcal{S}_n \times \mathcal{S}_2^n)\Big|$, $\forall\, g, f$:

$$\mathcal{S}_n = \{\,(\sigma(1),\dots,\sigma(2n)),\ \dots,\ (\sigma(2n-1),\sigma(2n),\dots,\sigma(1),\sigma(2))\,\}$$
$$\mathcal{S}_2^n = \{\,(\sigma(1),\dots,\sigma(2n)),\ \dots,\ (\sigma(2),\sigma(1),\dots,\sigma(2n),\sigma(2n-1))\,\}$$

and, $\mathcal{S}_n \times \mathcal{S}_2^n \cong$ equivalence classes $[\sigma]$, $\forall\,\sigma \in \mathrm{Aut}(\mathscr{D})$, by *bijection*. □

Lemma (orientation-parity ρ). *Let $\rho =$ cycle orientation parity, such that $\Gamma =$ quadratic lattice; if $\rho =$ odd for all mesh, then $\rho =$ odd (resp. even) for arbitrary cycle of even (resp. odd) vertices.*

Proof. ♡.

Corollary. *Monomials of* $D_1, D_2 \subseteq \mathscr{D}$ *have equal sign* $\Longleftrightarrow$ *orientation parity is odd for all transition cycles of* $D_1 \Delta D_2 = D_1 \cup D_2 \setminus D_1 \cap D_2$, *where sign of monomial of* D *is given by*

$$\varepsilon_D^K \;=\; (-1)^{t(\sigma)} \prod_{\ell=1}^{n} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \quad \Big| \ell \in D \subseteq \mathscr{D},\ \forall\, \sigma \in \mathsf{Aut}(D).$$

Proof. ε_D^K is automorphism-invariant; let $\ell, \ell' = 1, \ldots, n$; for partitions σ, τ of D_1, D_2 respectively matching $\sigma(2\ell-1)$ to $\sigma(2\ell)$, $\tau(2\ell'-1)$ to $\tau(2\ell')$:

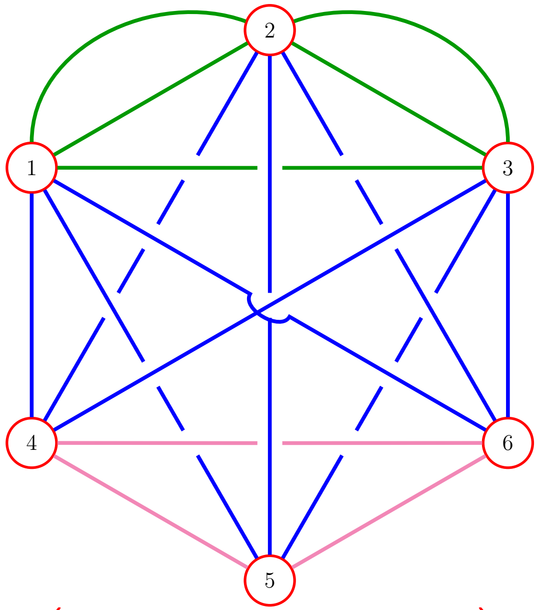
$$\varepsilon_{D_1}^K \varepsilon_{D_2}^K \;=\; (-1)^{t(\widetilde{\gamma})} \prod_{\ell=1}^{n} \varepsilon^K_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \prod_{\ell'=1}^{n} \varepsilon^K_{\widetilde{\tau}(2\ell'-1)\widetilde{\tau}(2\ell')} \quad \Big| \widetilde{\gamma} = \widetilde{\sigma} \circ \widetilde{\tau}$$

$$= \prod_{\ell=1}^{n} \varepsilon^K_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \prod_{\ell'=1}^{n} \varepsilon^K_{\widetilde{\tau}(2\ell'-1)\widetilde{\tau}(2\ell')} \;=\; (-1)^{\left(\sum_{C_\alpha} \sigma_{C_\alpha}(D_1 \Delta D_2)\right)} \quad \Big| \text{by } \rho \text{ lemma}$$

$\Longleftrightarrow \forall\, \eta = \sum_{C_\alpha} \sigma_{C_\alpha}(D_1 \Delta D_2)$, $\; n^K_{C_\alpha}\big|_{\mathscr{D}} \pmod{\eta} = \text{odd}$ is given by

$$\Big(\sum_{C_\alpha} \sum_{\ell \in \mathscr{D}} \mathbb{1}_{\left\{\varepsilon(\partial \mathcal{F}; i_\ell, j_\ell) \,=\, \varepsilon^K(\ell; j_\ell, i_\ell)\right\}}(C_\alpha) \Big) \pmod{\eta}$$

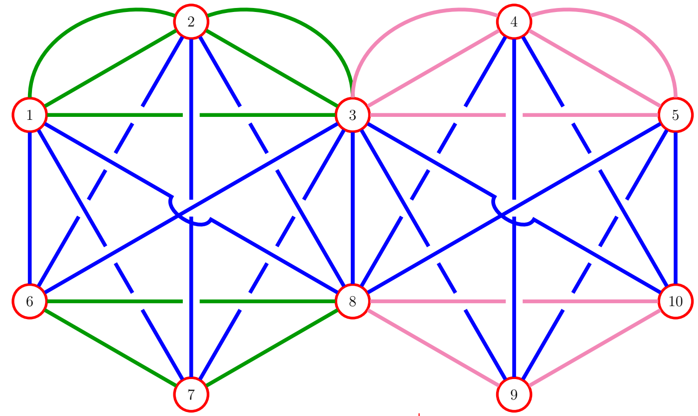
$\Longleftrightarrow n^K_{C_\alpha}\big|_{C_\alpha} = \text{odd}$, for all transition cycles $C_\alpha \mid \alpha = 1, \ldots, \eta$. $\square$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

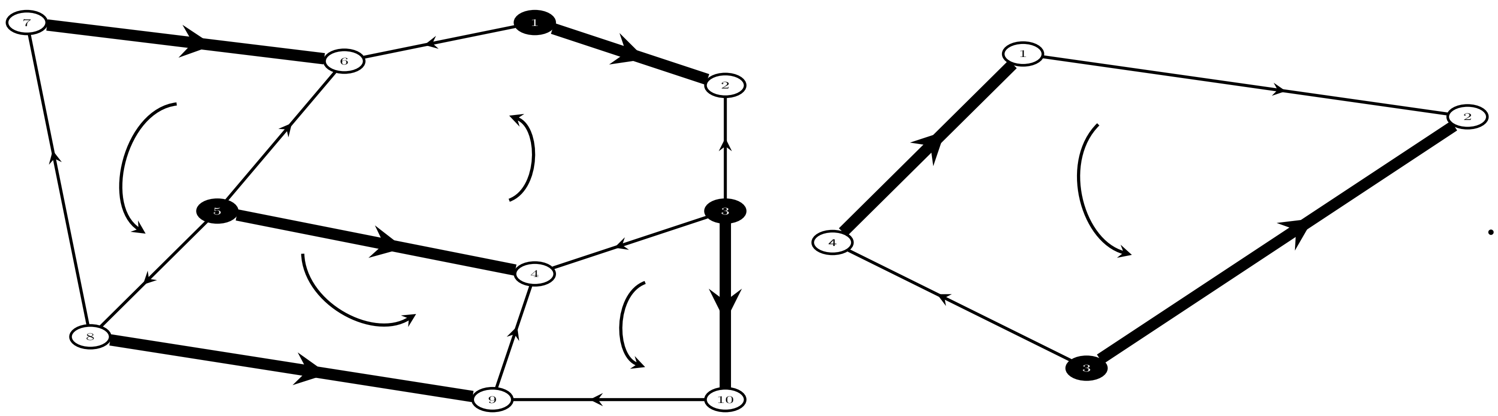
$\mathbf{0}$ = non-adjoined (i, j)

$\mathbf{1}, \mathbf{1}, \mathbf{1}$ = adjoined (i, j).



$$\left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

1.4 Kasteleyn orientation and matrix



Definition. Let $\Gamma \subset \overline{\mathcal{M}}_g = 1$-skeleton CW-complex (resp. genus g compact orientable surface cell-decomposition) embedding. Set of arbitrary orientation $\varepsilon^K(\ell;\cdot,\cdot)$, $\forall\, \ell \in \Gamma$, is Kasteleyn if with respect to the fixed (counterclockwise) i_ℓ to j_ℓ boundary orientation $\varepsilon(\partial\mathcal{F}; i_\ell, j_\ell)$, $\forall\, i_\ell \neq j_\ell$ faces $\mathcal{F}$ of edges ℓ,

$$\prod_{\ell \in \partial\mathcal{F}} \varepsilon^K_{i_\ell j_\ell} = -1, \ \ \forall\, \mathcal{F} \in \Gamma \quad \Bigg| \varepsilon^K_{i_\ell j_\ell} = \begin{cases} -1 & \text{if } \ \varepsilon(\partial\mathcal{F}; i_\ell, j_\ell) \ = \ \varepsilon^K(\ell; j_\ell, i_\ell) \\ +1 & \text{if } \ \varepsilon(\partial\mathcal{F}; i_\ell, j_\ell) \ = \ \varepsilon^K(\ell; i_\ell, j_\ell). \end{cases}$$

If $\Gamma =$ Kasteleyn, weighted $\omega_{i_\ell j_\ell} = \omega_{j_\ell i_\ell} > 0$, $\forall\, \{i_\ell \neq j_\ell\}$ edges ℓ, then

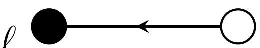
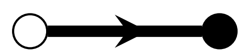
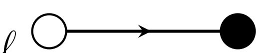
$$\Gamma^K_{ij} = \sum_{\ell} \varepsilon^K_{i_\ell j_\ell}\, \omega_{i_\ell j_\ell} \ = -\Gamma^K_{ji} \quad \Bigg| \Gamma^K_{ij} = 0, \ \ \forall\ i_\ell, j_{\ell'} \,|\, \ell \neq \ell' \text{ or } \forall\ i = j.$$

Remark.

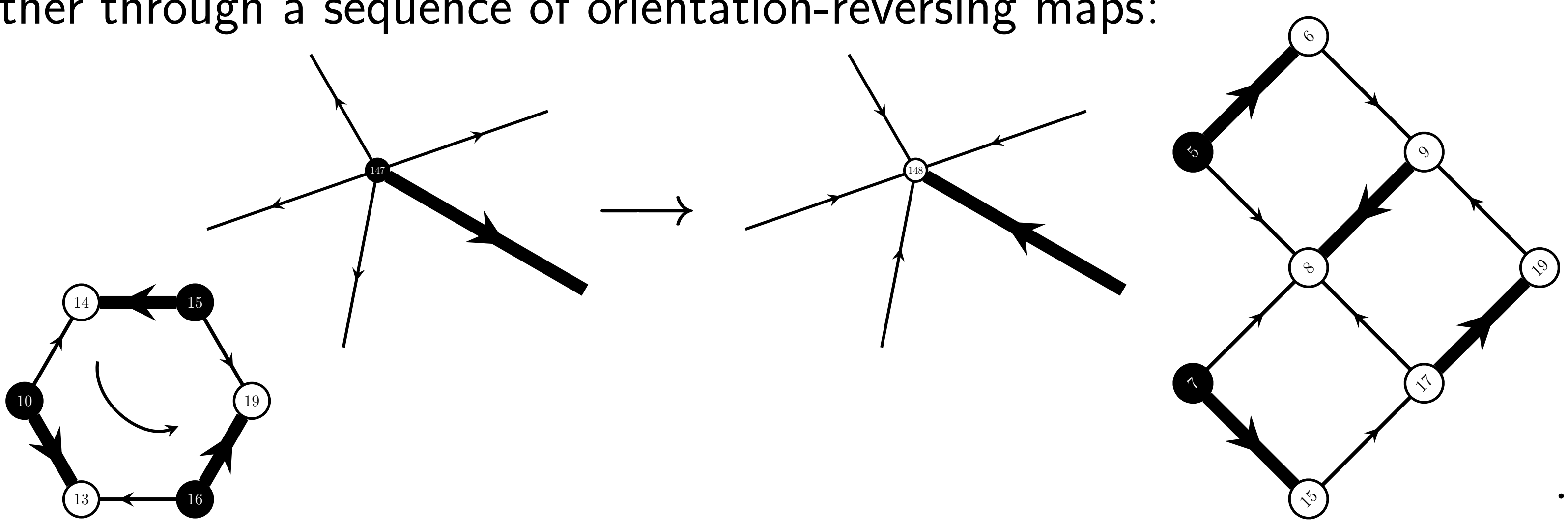
$$(\Gamma^K_{ij}) = \begin{cases} \text{Adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} = 1, \ \forall\, \varepsilon^K_{i_\ell j_\ell} = 1 \\ \text{Weighted adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} \neq 1, \ \forall\, \varepsilon^K_{i_\ell j_\ell} = 1 \\ \text{Skew-symmetric adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} = 1, \ \forall\, \varepsilon^K_{i_\ell j_\ell} = \pm 1. \end{cases}$$

Derivation (Kasteleyn). For counterclockwise $\varepsilon(\partial\mathcal{F})$ bipartite $2n$ lattice:

$\Gamma^K_{ij} = -\Gamma^K_{ji} =$

- $-\omega_{ij} = -\omega_{ji}$ if i_ℓ  j_ℓ or i_ℓ  j_ℓ
- $\omega_{ij} = \omega_{ji}$ if i_ℓ  j_ℓ or i_ℓ  j_ℓ
- 0 if $i = j$ or $\ell \neq \ell', \forall\, i_\ell, j_{\ell'}$.

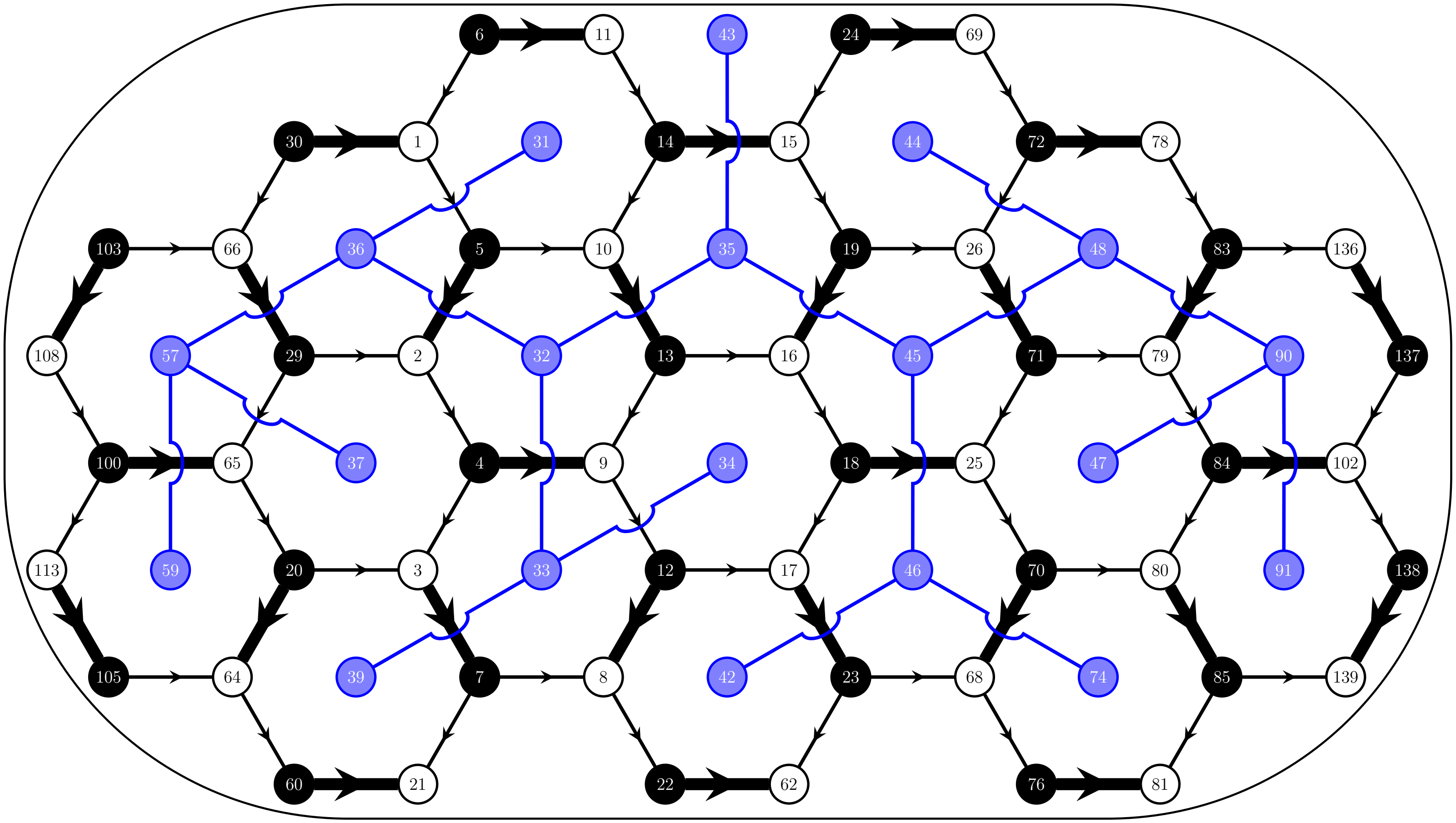
Definition. Two orientations are equivalent, if each is obtainable from the other through a sequence of orientation-reversing maps:



.

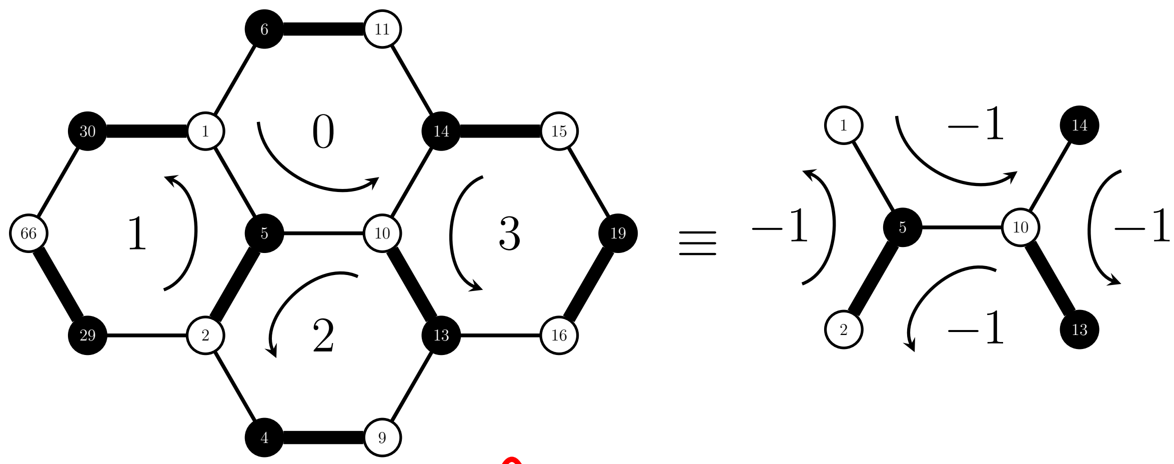
Lemma (existence). *Kasteleyn orientation exists.*

Proof. Following dual Γ^* spanning tree:

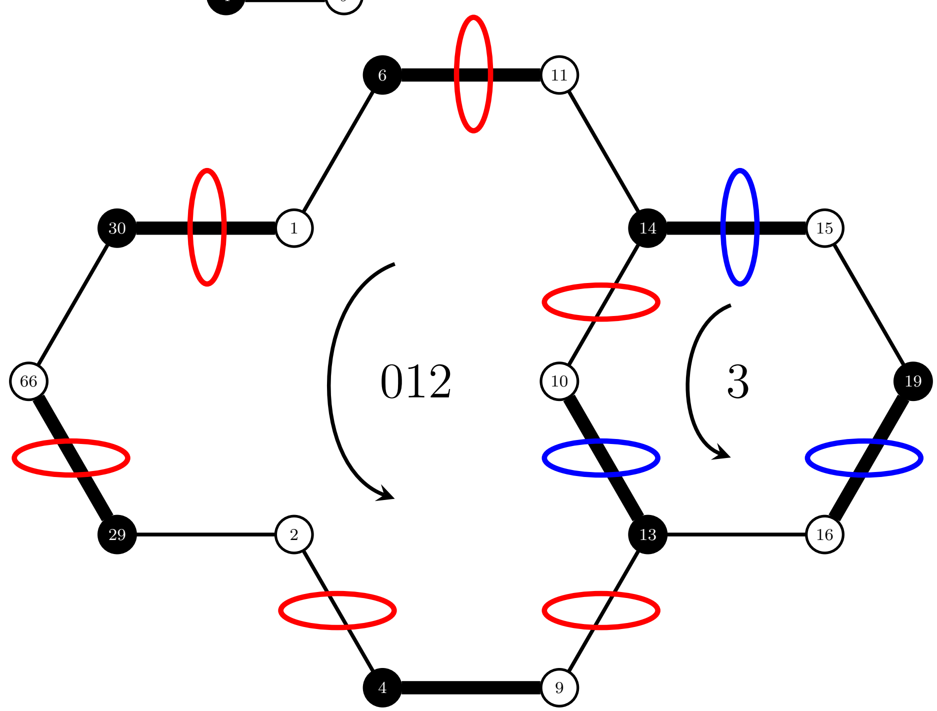


Reduce Γ to $\ll |\, n\times n \longrightarrow \exp(\alpha n^2)$. Arbitrarily orient all $\ell \in \Gamma$ not crossing Γ^* spanning tree rooted outside Γ. Deleting ℓ^* from leaves, make $\varepsilon^K(\mathcal{F})$.

Remark. Deleted-vertex changes Kasteleyn to non-Kasteleyn at "hole":



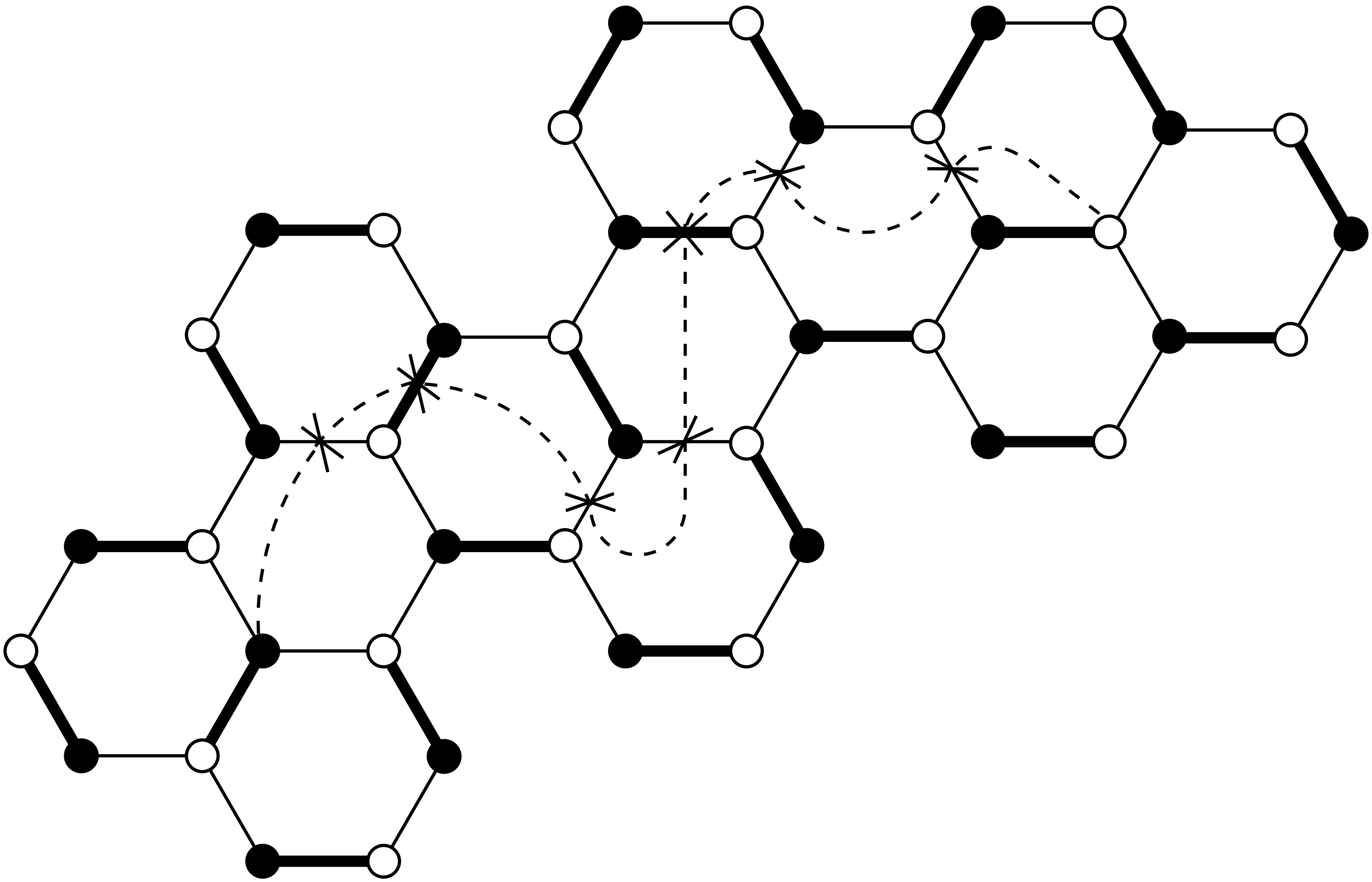
$h_0 = h_1 =$
$= h_2 = h_3 =$
$=$ Kasteleyn.



$h_{012} =$ non-Kasteleyn.

$h_3 =$ Kasteleyn.

Remark. To convert the non-Kasteleyn orientation back to Kasteleyn:



$$h_0 = h_1 = \cdots = h_{11} = -1.$$

Lemma. *All Kasteleyn orientations are equivalent for Γ planar.*

Proof. Consider arbitrary Kasteleyn orientations K and K': marked by K on ith end ℓ_-, K' on jth end ℓ_+, $\forall\,\ell\in\Gamma\subset\overline{\mathcal{M}}_g$, with respect to $\varepsilon(\partial\mathcal{F};i_\ell,j_\ell)$.

The product

$$\prod_{\ell\,\in\,\partial\mathcal{F}}\varepsilon^{KK'}_{i_\ell j_\ell}=\prod_{\ell_-\,\in\,\partial\mathcal{F}}\varepsilon^{K}_{i_{\ell_-}j_{\ell_-}}\cdot\prod_{\ell_+\,\in\,\partial\mathcal{F}}\varepsilon^{K'}_{i_{\ell_+}j_{\ell_+}}=\prod_{\ell_-,\,\ell_+\,\in\,\partial\mathcal{F}}\varepsilon^{K}_{i_{\ell_-}j_{\ell_-}}\cdot\varepsilon^{K'}_{i_{\ell_+}j_{\ell_+}}$$

which equals

$$\prod_{\ell_-\,\in\,\partial\mathcal{F}}\varepsilon^{K}_{i_{\ell_-}j_{\ell_-}}\cdot\prod_{\ell_+,\,\ell_-\,\in\,\partial\mathcal{F}}\sigma^{KK'}_{\ell_+}\cdot\varepsilon^{K}_{i_{\ell_-}j_{\ell_-}}\;=\;(-1)\cdot(-1)\;=\;1\quad\Big|\,\sigma^{KK'}_{\ell_+}\;=\;\pm1$$

where

$$\varepsilon^{KK'}_{i_\ell j_\ell}=\begin{cases}-1 & \text{if}\;\;\varepsilon^{K}(\ell_-;i_{\ell_-},j_{\ell_-})\;=\;\varepsilon^{K'}(\ell_+;j_{\ell_+},i_{\ell_+})\;\;\text{i.e.}\;\;\sigma^{KK'}_{\ell_+}\;=\;-1\\ +1 & \text{if}\;\;\varepsilon^{K}(\ell_-;i_{\ell_-},j_{\ell_-})\;=\;\varepsilon^{K'}(\ell_+;i_{\ell_+},j_{\ell_+})\;\;\text{i.e.}\;\;\sigma^{KK'}_{\ell_+}\;=\;+1\end{cases}$$

is well-defined for all K from K' and vice versa $K\longleftrightarrow K'$ in simple reversal of orientations around vertices, as required, by $\sigma^{KK'}_{\ell_+}=-1$. □

Lemma. *Equivalence class $[K]$ is unique for Γ planar.*

Proof. Follows from fundamental group being trivial for the plane. □

Theorem. Equivalence classes $[K]$ of Kasteleyn orientations equal 2^{2g}.

Proof. The equivalence classes $\{[K]\} \cong$ affine closure of non-degenerate skew-symmetric, quadratic form $q \in \mathsf{Sym}^2_k(V^\wedge)$ on characteristic-2 field k:

$$q(\alpha+\beta) = q(\alpha) + q(\beta) + \alpha\cdot\beta \ \Big|\, q: V\otimes V \longrightarrow k\,,\ \forall\,\alpha,\beta \in \mathcal{H}^1 = V\otimes V$$

$\forall\,\alpha \in \mathcal{H}^1 =$ first homology space, classified by:

$$\frac{1}{\sqrt{|\mathcal{H}^1|}} \sum_{q\,\in\,(\mathcal{H}^1,\cdot)} (-1)^{\mathit{Arf}(q)+q(\alpha)} = 1 \ \Big|\, \mathit{Arf}(q) = \sum_{\{e_i,\,e_j\}} q(e_i)q(e_j) \in (k/f(k)) \subset \mathbb{Z}_2$$

where $\{e_i, e_j\} =$ symplectic basis-pairs in symplectomorphisms $V \longrightarrow V$, for Lang's isogeny $f: k \longrightarrow k\ |\, x \longmapsto x^2 - x \in \mathbb{G}_a/\mathbb{F}_2$.

But, continuous $\psi: \Gamma \longrightarrow \overline{\mathcal{M}}_g \mid \Gamma \supseteq \mathcal{F} = \psi$-faces $\approx$ open disk $=$ connected components of $\overline{\mathcal{M}}_g \backslash \psi(\Gamma)$ implies $\chi(\Gamma) = \chi(\overline{\mathcal{M}}_g)$ in Euler-Poincaré bound $|V| - |E| + |\mathcal{F}| = \chi(\Gamma) \geqslant \chi(\overline{\mathcal{M}}_g)$. And, all vanishing composition $\partial_1 \circ \partial_2$ of boundary operators $\partial_2: C_2 \longrightarrow C_1$, $\partial_1: C_1 \longrightarrow C_0$, $\forall\, C_0, C_1, C_2 =$ vertex-, edge-, face-basis of 2D cell-complex, implies 1-cycle space $\mathsf{Ker}(\partial_1)$ contains 1-boundary space $\partial_2(C_2)$. Hence, $|\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)| = |\mathsf{Ker}(\partial_1)/\partial_2(C_2)| = 2g$, depending only on the genus g, independent of Γ. $\square$

Theorem (Kasteleyn). *Let* $\Gamma = (i_\ell \mid \ell' \geqslant \ell \in \mathbb{N}^+,\, i_{\ell'} \neq i_\ell) \subset \overline{\mathcal{M}}_g \mid g \gg$ *be embedding for all equivalence classes* $[\sigma]$ *of perfect matchings* $\mathscr{D} \supseteq D$, $\forall\, \sigma \in$ *Aut*$(\mathscr{D})$, *sgn*$(\sigma) = (-1)^{t(\sigma)} \,|\, t(\sigma) := (\sigma(1), \ldots, \sigma(2n)) \longrightarrow (1, \ldots, 2n)$; *the partition function* $Z = \sum_{D \subset \mathscr{D}} \prod_{\ell \in D} \varepsilon_\ell^K \omega_\ell = \pm\, \mathrm{Pf}(\Gamma^K) \in \mathbf{Quot}(\mathbb{K}[D])$ *is given by*

$$\text{(i)} \quad \mathrm{Pf}(\Gamma^K) \;=\; \frac{1}{n!}\,\frac{1}{2^n} \sum_{\sigma \in \mathcal{S}_{2n}} (-1)^{t(\sigma)} \prod_{\ell=1}^{n} \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)}$$

$$\text{(ii)} \quad \mathrm{Pf}(\Gamma^K) \;=\; \sum_{\sigma \big|_{[\sigma]}} (-1)^{t(\sigma)} \prod_{\ell=1}^{n} \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)}\,.$$

Proof. (i)$\Longrightarrow$(ii) by perfect-matching *bijection*. To see (i): $\Gamma^K = m\times m \Longrightarrow \det\Gamma^K = \det(-(\Gamma^K)^T) = (-1)^m \det\Gamma^K = 0 \Longleftrightarrow m = \text{odd}$; $\det\Gamma^K \neq 0 \Longrightarrow \det\Gamma^K =$ positive-definite, square of rational function of $\Gamma^K_{ij} \mid \Gamma^K = 2n\times 2n$.

In particular, $\Gamma^K_{i\,\rho(i)} = -\Gamma^K_{\rho(i)\,i} \mid i \leqslant \rho(i) \Longrightarrow$ sum of 2-partition monomials:

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{\rho \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n)}} (-1)^{t(\rho)} \prod_{i=1}^{2n} \Gamma^K_{i\,\rho(i)} \quad \left| \begin{array}{l} j = \rho^{-1}(i) \longleftrightarrow i \neq j \in \{1,\ldots,n\} \\ \quad\Longrightarrow \Gamma^K_{i\,\rho(i)} \equiv \Gamma^K_{\rho(2\ell-1)\rho(2\ell)} \\ \quad\forall\, \ell = 1,\ldots,2n; \\ t(\rho) = \text{even (odd), for even } 2n \text{ (otherwise)} \\ t(\rho) := (\rho(1),\ldots,\rho(2n)) \longrightarrow (1,\ldots,2n) \end{array}\right. \\ \qquad\qquad + \\ \displaystyle\sum_{\substack{\rho \\ \Cap \\ 2\cdot\left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}} \Big/ \left(\mathcal{S}_2\times\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}-2\right\}}\right)\right)}} (-1)^{t(\rho)} \prod_{i=1}^{2n} \Gamma^K_{i\,\rho(i)} \quad \left| \begin{array}{l} j = \rho^{-1}(i) \longleftrightarrow i \neq j \in \{1,\ldots,n\} \\ \quad\Longrightarrow \Gamma^K_{i\,\rho(i)} \equiv \Gamma^K_{\rho(2\ell-1)\rho(2\ell)} \\ \quad\forall\, \ell = 1,\ldots,2n; \\ t(\rho) = \text{odd (even)}, \\ \text{for even } 2n \text{ (otherwise).} \end{array}\right. \end{array}\right.$$

by Leibniz's second-index permutations.

Evidently, $\Gamma^K_{i\,\rho(i)} = -\Gamma^K_{\rho(i)\,i} \mid i \leqslant \rho(i) \Longrightarrow$ the quadratic multinomial:

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{\sigma=\widetilde{\sigma} \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n)}} (-1)^{t(\rho)+n+t(\sigma)} \left(\prod_{\ell=1}^n \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)}\right)^2 \quad \Bigg| \begin{array}{l} t(\rho) = \text{even (odd)}, \\ \text{for even } 2n \text{ (otherwise)} \end{array} \\ \qquad\qquad + \\ 2\times \displaystyle\sum_{\substack{\left(\begin{array}{c}\sigma=\widetilde{\sigma}\neq\tau=\widetilde{\tau} \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n)\end{array}\right) \\ \cong \\ \left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}}\Big/\left(\mathcal{S}_2\times\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}-2\right\}}\right)\right)}} (-1)^{t(\sigma)+t(\tau)} \prod_{\ell=1}^n \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)} \prod_{\ell=1}^n \Gamma^K_{\tau(2\ell-1)\tau(2\ell)} \end{array}\right.$$

$$= \left(\sum_{\sigma=\widetilde{\sigma}} (-1)^{t(\sigma)} \prod_{\ell=1}^n \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)}\right)^2 = \mathsf{Pf}^2(\Gamma^K) \;\Bigg|\; \begin{array}{r} t(\sigma) := (\sigma(1),\ldots,\sigma(2n)) \\ \longrightarrow\; (1,\ldots,2n). \end{array}$$

$$\forall\ \min(\deg(\Gamma)) \quad \Big|\ \{[\sigma]\} \subseteq \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n).$$

Now, write Z by all non-vanishing monomials $\forall\, D$:

$$Z \;=\; \sum_{\sigma=\widetilde{\sigma}} \sum_{D} \prod_{\ell=1}^{n} \varepsilon^K_{\sigma^D(2\ell-1)\sigma^D(2\ell)}\; \omega_{\sigma^D(2\ell-1)\sigma^D(2\ell)}$$

where, if D and $\widetilde{\sigma}$ are 1-1, then $Z = \sum_{\sigma=\widetilde{\sigma}} \prod_{\ell=1}^{n} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \omega_{\sigma(2\ell-1)\sigma(2\ell)}$.

But, for any $\{\sigma\,\big|_{[\sigma]}\}$ = set of all partitions of disjoint equivalence classes:

$$\mathrm{Pf}(\Gamma^K) = \sum_{\sigma|_{[\sigma]}} \underbrace{(-1)^{t(\widetilde{\sigma})-\Delta(\sigma)} \prod_{\ell=1}^{n} \Gamma^K_{\sigma(2\ell-1)\sigma(2\ell)}}_{D\text{-fixed }\pm,\ \forall\,\sigma|_{[\sigma]}} \;\left|\; \begin{array}{l} \Delta(\sigma) := \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \\ \qquad\qquad \longrightarrow \varepsilon^K_{\sigma(2\ell)\sigma(2\ell-1)} \\ \forall\, \sigma(2\ell-1) > \sigma(2\ell) \end{array}\right.$$

$$= \left\{ \frac{1}{n!}\frac{1}{2^n} \sum_{\substack{\sigma \\ \cap \\ \mathrm{Aut}(\mathscr{D})}} \sum_{D} \overbrace{(-1)^{t(\sigma^D)} \prod_{\ell=1}^{n} \varepsilon^K_{\sigma^D(2\ell-1)\sigma^D(2\ell)}}^{D\text{-fixed }\pm,\ \forall\,\sigma^D|_{[\sigma^D]}} \prod_{\ell=1}^{n} \omega_{\sigma^D(2\ell-1)\sigma^D(2\ell)}. \right.$$

That is, such that all $\mathcal{S}_{2n}\backslash \mathrm{Aut}(\mathscr{D})$ monomials vanish, write:

$$\sum_{D\subseteq\mathscr{D}}\prod_{\ell\in D}\varepsilon_\ell^K\omega_\ell = \quad \Big|\ \text{the } \textit{bijection } (-1)^{t(\widetilde{\sigma}^D)} = (-1)^{t(\sigma^D)+\Delta(\sigma^D)} \text{ equals}$$

$$= \mathrm{Pf}(\Gamma^K), \quad \Big|\ \frac{1}{n!}\frac{1}{2^n}\sum_{\sigma^D\in(\mathcal{S}_n\times\mathcal{S}_2^n)\big|_{[\sigma^D]}}(-1)^{t(\sigma^D)}\prod_{\ell=1}^{n}\varepsilon^K_{\sigma^D(2\ell-1)\sigma(2\ell)}$$

by (i) and (ii)

i.e. depending only on perfect-matching orientation; independent of $\widetilde{\sigma}^D$. $\square$

Corollary. *Correlation = Pfaffian of inverse Kasteleyn operator*

$$\left\langle\prod_{i=1}^{k}\sigma_D(\ell_i)\right\rangle = \mathrm{Pf}\big((\Gamma^K)^{-1}_{\xi\eta}\big) \quad \Big|\ \xi,\eta = 1,\dots,k; \quad \mathrm{Pf}(\Gamma^K) = \textit{partition function}.$$

Proof. Follows from Kasteleyn theorem for $n = k$. $\square$

Remark. Combinatorial (exponential) complexity reduces to cubic complexity of diagonalizing $\mathrm{Pf}(\mathcal{A}\,\Gamma^K\mathcal{A}^T)=\det(\mathcal{A})\mathrm{Pf}(\Gamma^K)\longrightarrow\mathcal{O}(n^3)$ in skew symmetric Gaussian elimination. And, the discrete property, correlation, is universal $\forall\, g$ since behavior of local observable is determined at point.

1.5 Computation of rank of equivalence classes

Rank $|\{\widetilde{\sigma}(\Gamma;\ \cdot,\cdot)\}|$ of equivalence classes $[\sigma]$ of all perfect matchings $\mathscr{D} \supseteq D$, for a fixed genus g domain, $\forall\, \sigma \in \mathrm{Aut}(\mathscr{D})$, is given by partition function i.e. matching polynomial: two-variable $z_1 = z_2 = 1$ generating function

$$\sum_{D(N_1,\ldots,N_k)\ |\ (\sum_{\nu=1}^{k} N_\nu) = n} (\pm) \prod_{\nu=1}^{k} \omega_\nu^{N_\nu} \quad \Big| N_\nu = \ |\text{edges}|\ \text{ of } \ z_\nu\text{-class.}$$

Derivation I. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ planar $M \times N$ square grid, where $\partial\Gamma =$ open.

$$|\{\widetilde{\sigma}(\Gamma;\ M, N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\cos^2\left(\frac{\pi i}{M+1}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| N = \text{even}$$

$$= |\{\widetilde{\sigma}(\Gamma;\ N, M)\}| \quad \Big| \begin{matrix} M = \text{even} \\ N = \text{odd} \end{matrix}$$

$$= 0 \quad \Big| MN = \text{odd.}$$

Show. ♡.

Derivation II. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ cylindrical $M \times N$ square grid.

$$|\{\widetilde{\sigma}(\Gamma;\, M, N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| N = \text{even}$$

$$= 2^{(\frac{MN}{2} - \frac{M}{2} + 1)} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| \begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. ♡.

Derivation III. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ toroidal $M \times N$ square grid.

$$|\{\widetilde{\boldsymbol{\sigma}}(\Gamma;\, M, N)\}| =$$

$$= 2^{(\frac{MN}{2}-1)} \left(\begin{array}{c} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\,(\frac{\pi(2i-1)}{M}) + \sin^2\,(\frac{2\pi j}{N})} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\,(\frac{2\pi i}{M}) + \sin^2\,(\frac{\pi(2j-1)}{N})} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\,(\frac{\pi(2i-1)}{M}) + \sin^2\,(\frac{\pi(2j-1)}{N})} \end{array} \right) \Bigg| \, N = \text{even}$$

$$= |\widetilde{\boldsymbol{\sigma}}(\Gamma;\, N, M)| \quad \Bigg| \begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. ♡.

Derivation IV. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ planar 6×8 square grid, where $\partial\Gamma =$ open.

$$|\{\widetilde{\sigma}(\Gamma;\, M, N)\}| =$$

$$= 16777216\,\Big(\frac{1}{4} + \cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{9}) + \cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{7}) + \cos^2(\frac{2\pi}{9})\Big) \times$$

$$\times\,\Big(\cos^2(\frac{\pi}{7}) + \sin^2(\frac{\pi}{18})\Big)\Big(\frac{1}{4} + \sin^2(\frac{\pi}{14})\Big)\Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{\pi}{14})\Big) \times$$

$$\times\,\Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{\pi}{14})\Big)\Big(\frac{1}{4} + \sin^2(\frac{3\pi}{14})\Big) \times$$

$$\times\,\Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big)\Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{3\pi}{14})\Big).$$

Show. ♡.

Derivation V. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ cylindrical 6×8 square grid.

$$|\{\widetilde{\sigma}(\Gamma;\ M, N)\}| =$$

$$= 5242880\,\Big(\frac{1}{4} + \cos^2(\frac{\pi}{9})\Big)^2\Big(1 + \cos^2(\frac{\pi}{9})\Big)\Big(\frac{1}{4} + \cos^2(\frac{2\pi}{9})\Big)^2 \times$$

$$\times\ \Big(1 + \cos^2(\frac{2\pi}{9})\Big)\Big(\frac{1}{4} + \sin^2(\frac{\pi}{18})\Big)^2\Big(1 + \sin^2(\frac{\pi}{18})\Big)$$

Show. $\heartsuit$.

Derivation VI. Let $\Gamma \subset \overline{\mathcal{M}}_g =$ toroidal 6×8 square grid.

$$|\{\widetilde{\sigma}(\Gamma;\ M, N)\}| =$$

$$= 8388608\,\Big[\,\frac{18225}{131072}\ +\ \cos^4(\frac{\pi}{8})\Big(\frac{3}{4} + \cos^2(\frac{\pi}{8})\Big)^4 \sin^4(\frac{\pi}{8})\Big(\frac{3}{4} + \sin^2(\frac{\pi}{8})\Big)^4 +$$

$$+\ \Big(\frac{1}{4} + \cos^2(\frac{\pi}{8})\Big)^4\Big(1 + \cos^2(\frac{\pi}{8})\Big)^2\Big(\frac{1}{4} + \sin^2(\frac{\pi}{8})\Big)^4\Big(1 + \sin^2(\frac{\pi}{8})\Big)^2\Big].$$

Show. $\heartsuit$.

1.6 Grassmann integral

Definition. The Grassmann algebra $\bigwedge^{\bullet} V$ on a basis $(a_1, \ldots, a_{2n})$ of V is generated by $2^{2n} = \sum_{k=0}^{2n} (\dim \bigwedge^k V) = \sum_{k=0}^{2n} \binom{2n}{k}$ dimensional basis vectors

$$\left\{ \begin{array}{l} a_0 = 1;\ a_{\sigma(k)<} = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},\ \forall\, \sigma(k)< \ = (\sigma(1), ..., \sigma(k)) \mid \sigma(1) < \cdots < \sigma(k); \\ a_{\sigma(\xi)} \otimes a_{\sigma(\eta)} + a_{\sigma(\eta)} \otimes a_{\sigma(\xi)} = 0, \forall\, \xi, \eta = 1, \ldots, k;\ \sigma(\xi), \sigma(\eta), k = 1, \ldots, 2n \end{array} \right\}$$

with:

(i) Element, k-component $(\forall\, k = 0, 1, \ldots, 2n)$

$$\bigwedge^k V \colon \bigotimes^k V \longrightarrow \bigotimes^k V \ \left| \begin{array}{l} \left(v^{\sigma(k)<} = \dfrac{1}{k!} \displaystyle\sum_{\sigma(k)<} \sum_{\sigma} (-1)^{t(\sigma)} \prod_{i=1}^{k} v_{i\,\sigma(i)} \right) a_{\sigma(k)<} \\ \sigma \in \mathcal{S}_{\sigma(k)<}\ ,\ \ t(\sigma) := (\sigma(1), ..., \sigma(k)) \longrightarrow \sigma(k)< \end{array} \right.$$

(ii) Multiplication $(\forall\, k, l = 0, 1, \ldots, 2n)$

$$\begin{array}{l} (vw)^{\sigma(k)<\ \sigma(l)<} = v^{\sigma(k)<} w^{\sigma(l)<} \\ (vw)^0 = v_0 w_0 \end{array} \ \left| \begin{array}{l} \sigma(i)\big|_{\sigma(k)<} = \sigma(j)\big|_{\sigma(l)<} \Longrightarrow 0 \\ \forall\, \bigwedge^k V,\ \bigwedge^l V;\ k, l = 1, \ldots, 2n \end{array} \right.$$

$$\forall\, v \in \mathrm{Span}(\textstyle\bigwedge^{\bullet} V) \colon \left\{ v = v_0 + \displaystyle\sum_{k=1}^{2n} v_k\, a_k + \sum_{k=2}^{2n} v^{\sigma(k)<} a_{\sigma(k)<} \ \middle|\ v_{(\cdot)} \in \mathbb{C} \right\}.$$

Definition. The $\bigwedge^{\bullet} V$ integral, with respect to orientation $\mathbb{R} \simeq x \in \bigwedge^{2n} V$:

$$\int_{\bigwedge^{2n} V} f \;=\; f_x \quad \Big|\, f = f_x\, x + \underbrace{\cdots}_{\text{lower order terms}}$$

where if (a_i) is a basis in V, then $x = a_1 \otimes \cdots \otimes a_n$ by the formal constraints:

$$\text{(i)} \int \Big(\bigotimes_{i=1}^{k} a_{\boldsymbol{\sigma}(i)} \Big) \otimes da = \begin{cases} 0 & \big| k < 2n \\ (-1)^{t(\boldsymbol{\sigma})} & \big| k = 2n \end{cases} \quad \Bigg| \begin{array}{l} da \;\cong\; (-1)^{n(2n-1)} \bigotimes_{i=1}^{2n} da_i \\ t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}(1), \ldots, \boldsymbol{\sigma}(2n)) \\ \qquad \longrightarrow \; (1, \ldots, 2n). \end{array}$$

$$\text{(ii)} \int \Big(\bigotimes_{i=1}^{2n} a_i \Big) \otimes \Big(\bigotimes_{i=1}^{2n} da_i \Big) \;=\; (-1)^{n(2n-1)} \int \bigotimes_{i=1}^{2n} \Big(a_i \otimes da_i \Big) \;=\; (-1)^{n(2n-1)}.$$

Lemma. *$\bigwedge^{\bullet} V$ graded identity, up to tensors in superalgebra $M_{a,b}$ of minimal subfield, is isomorphic to kernel of either $\mathbb{Q}$ or prime-ordered field $\mathbb{F}_{q=p^m}$.*
Proof. $\heartsuit$.

Theorem. *T-ideal of $M_{pr+qs,\, ps+qr}$ is contained in T-ideal of $M_{p,q} \otimes M_{r,s}$.*
Proof. Follows from the prior lemma.

Theorem. *Let* $A^*(a) = \int_{\bigwedge^\bullet V} A(a) \; \big| A(a) = \exp\big(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\big),\ (a_i) \subseteq V,$ *satisfy the Grassmann constraints*; A^* *uniquely maximizes* $-\int_{\bigwedge^\bullet V} A \log A$ *over all* A *satisfying the integral such that*:

(i) $\mathsf{Pf}(A) = \int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\Big) da$

(ii) $\mathsf{Pf}\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = \det(A)$

(iii) $(\mathsf{Pf}(A))^2 = \det(A)$

(iv) $\frac{\partial}{\partial A_{i_1 j_1}} \cdots \frac{\partial}{\partial A_{i_k j_k}} \mathsf{Pf}(A) =$

$$= \mathsf{Pf}(A) \cdot \mathsf{Pf}((A^{-1})_{ab}) \; \Big|_{\substack{a = i_1, \ldots, i_k \\ b = j_1, \ldots, j_k}} .$$

Proof (hints).

(i). Write:

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big) da \quad = \quad \frac{1}{n!}\frac{1}{2^n} \int_{\bigwedge^\bullet V} \langle a, Aa\rangle^n \, da$$

such that

$$\int \langle a, Aa\rangle^{2n} \, da \;=\; \int a_{\sigma(1)} a_{\tau(1)} \cdots a_{\sigma(n)} a_{\tau(n)} \, A_{\sigma(1)\,\tau(1)} \cdots A_{\sigma(n)\,\tau(n)} \; da \;=$$

$$= \;(-1)^{t(\sigma)} A_{\sigma(1)\,\tau(1)} \cdots A_{\sigma(n)\,\tau(n)} \qquad \Bigg|\, \begin{array}{l} t(\sigma): \;(\sigma(1), \tau(1), \ldots, \sigma(n), \tau(n)) \\ \qquad\qquad \longrightarrow\; (1, \ldots, 2n). \end{array}$$

This implies

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big) da \quad = \quad \frac{1}{n!}\frac{1}{2^n}\, \mathsf{Pf}(A).$$

Remark. The integral formula is used to prove II, III, IV. $\heartsuit$.

(ii). Choosing splitting $V = W \oplus W^*$ by matrix block structure, where V Grassmann algebra is isomorphic to algebra (tensor product) generated by $c_i, b_i \,|\, i = 1, \ldots, n$ with relations $c_i c_j = -c_j c_i$, $c_i b_j = -b_j c_i$ and $b_1 b_j = -b_j b_i$:

$$
\begin{aligned}
(a_1, \ldots, a_{2n}) &= \\
&= (\underbrace{c_1, \ldots, c_n}_{\text{basis in } W}, \underbrace{b_1, \ldots, b_n}_{\text{basis in } W^*}).
\end{aligned}
$$

As a result,

$$
\left\langle a \,, \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} a \right\rangle \;=\; 2 \left\langle c, Ab \right\rangle .
$$

Therefore, prove

$$
\int\limits_{\bigwedge^n(W \oplus W^*)} \exp\left(\langle c, Ab\rangle\right) dc\, db \;=\; \det(A).
$$

(iii). Similar.

(iv). $$\int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle a, \eta\rangle\Big)da \quad =$$

$$= \quad \int \exp\Big(\frac{1}{2}\langle a+A^{-1}\eta\,,\ A(a+A^{-1}\eta)\rangle\ -\ \frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big)da$$

$$= \quad \mathsf{Pf}(A)\ \exp\Big(-\frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big).$$

$$\frac{\partial}{\partial A_{i_1 j_1}} \cdots \frac{\partial}{\partial A_{i_k j_k}}\ \mathsf{Pf}(A) \quad =$$

$$= \quad \int a_{i_1} a_{j_1} \cdots a_{i_k} a_{j_k} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)da$$

$$= \quad \Big(\frac{\partial}{\partial \eta}\Big)^{2k} \int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle \eta, a\rangle\Big)da.$$

Theorem (bipartite). *Let* $\Gamma \subset \mathbb{R}^2 =$ *Kasteleyn-oriented bipartite partition*,

$$Z_\Gamma = \varepsilon_\Gamma^K \int \exp\Big(\frac{1}{2}\sum_{ij} a_i\,(\Gamma_{ij}^K)\,a_j\Big)\,da \;\left|\; \begin{array}{l} \varepsilon_\Gamma^K = (-1)^\sigma\,\varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{2n-1}\sigma_{2n}} \in \{\pm 1\} \\ 2n = |V(\Gamma)|. \end{array}\right.$$

Proof. $\Gamma =$ bipartite implies

$$\Gamma^K = \begin{pmatrix} 0 & B_\Gamma^K \\ -(B_\Gamma^K)^T & 0 \end{pmatrix} \;\left|\; \begin{array}{l} B^K : \mathbb{R}^{V_\circ(\Gamma)} \longrightarrow \mathbb{R}^{V_\bullet(\Gamma)} \\ \mathbb{R}^{V(\Gamma)} = \mathbb{R}^{V_\bullet(\Gamma)} \oplus \mathbb{R}^{V_\circ(\Gamma)} \\ \dim(\mathbb{R}^{V_\bullet(\Gamma)}) = \dim(\mathbb{R}^{V_\circ(\Gamma)}) = n \\ V(\Gamma) = V_\bullet(\Gamma) \sqcup V_\circ(\Gamma),\;\; |V(\Gamma)| = 2n. \end{array}\right.$$

Identifying $V_\bullet(\Gamma)$, $V_\circ(\Gamma)$ via a diagram $\{b\} \sim \{\omega\}$ with "hole"

$$\Gamma^K = \begin{pmatrix} 0 & C_\Gamma^K \\ -(C_\Gamma^K)^T & 0 \end{pmatrix} \;\left|\; \begin{array}{l} \mathbb{R}^{V(\Gamma)} = \mathbb{R}^{V_\bullet(\Gamma)} \oplus \mathbb{R}^{V_\circ(\Gamma)} \hookleftarrow \\ C_\Gamma^K = \mathbb{R}^{V_\circ(\Gamma)} \hookleftarrow \\ \hookleftarrow \Longrightarrow \text{ recursion i.e. nested.} \end{array}\right.$$

As a result, $Z_\Gamma = |\det(C_\Gamma^K)|$ $\square$

Corollary (bipartite correlation).

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle = \frac{\partial}{\partial\, \omega(b_1 w_1)} \cdots \frac{\partial}{\partial\, \omega(b_k w_k)} \ln Z_\Gamma$$

$$\Longrightarrow \langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle = \det\Big(\big((C_\Gamma^K)^{-1} \big)_{\widetilde{b}\, w} \Big) \quad \Bigg| \begin{array}{l} \widetilde{b} = \widetilde{b}_1, \ldots, \widetilde{b}_k \\ w = w_1, \ldots, w_k \end{array}$$

where $\widetilde{b}$ = white-vertex identified with b.

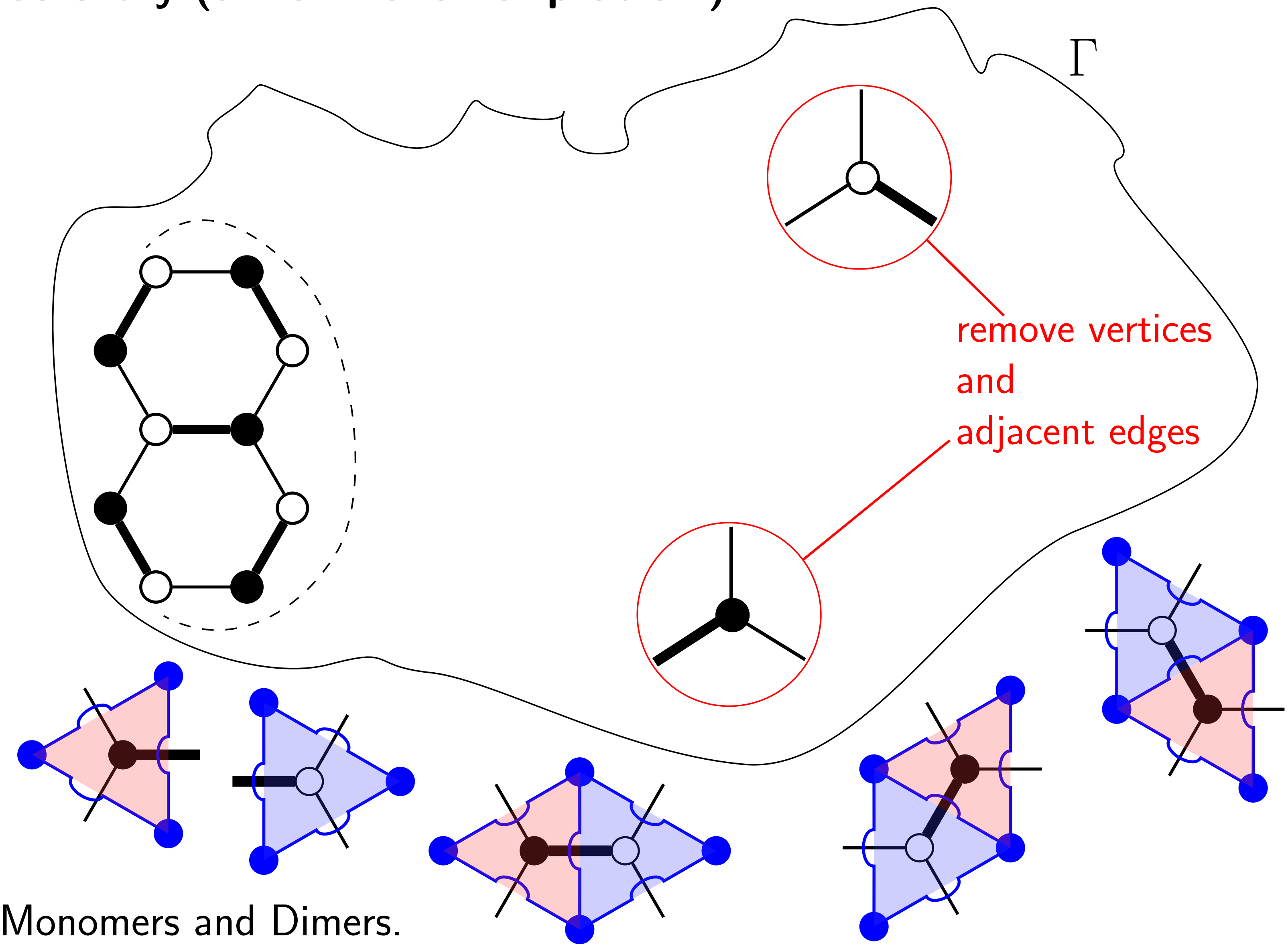
Remark. The "physical" meaning of (ii):

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle =$$

$$= \int \psi^*_{b_1} \psi_{w_1} \cdots \psi^*_{b_k} \psi_{w_k} \exp\big(\psi^* C_\Gamma^K \psi\big) d\psi^* d\psi \ \cdot \Big(\int \exp\big(\psi^* C_\Gamma^K \psi\big) d\psi^* d\psi \Big)$$

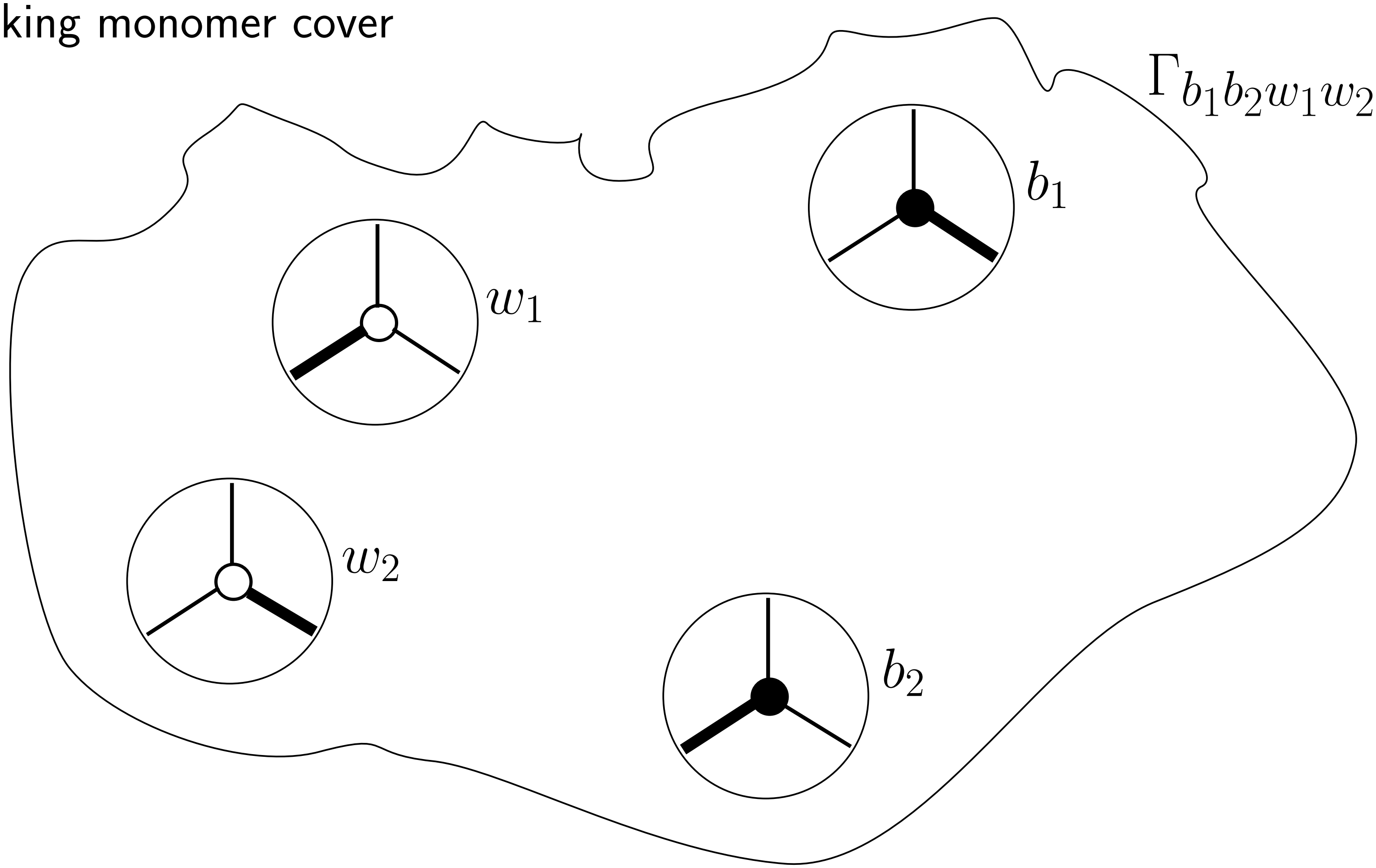
which corresponds to correlation for the free Fermionic representation.

Corollary (dimer-monomer problem).



Monomers and Dimers.

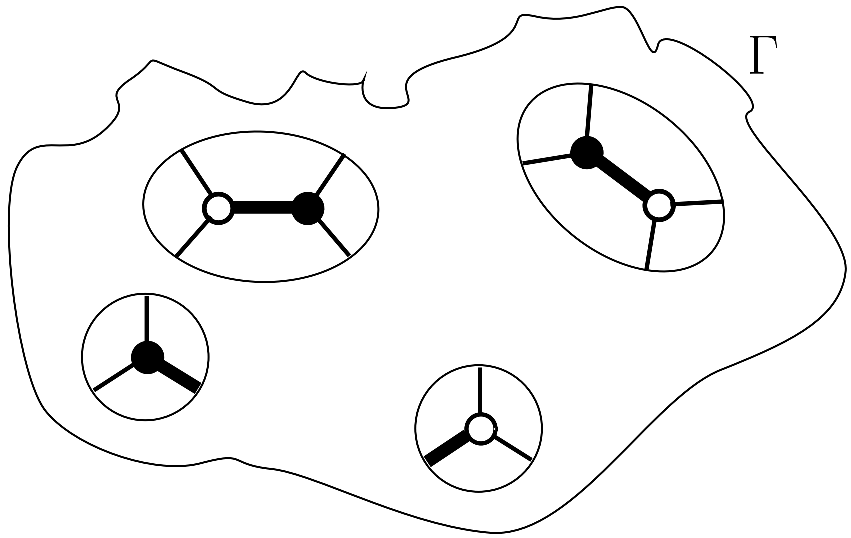
Taking monomer cover



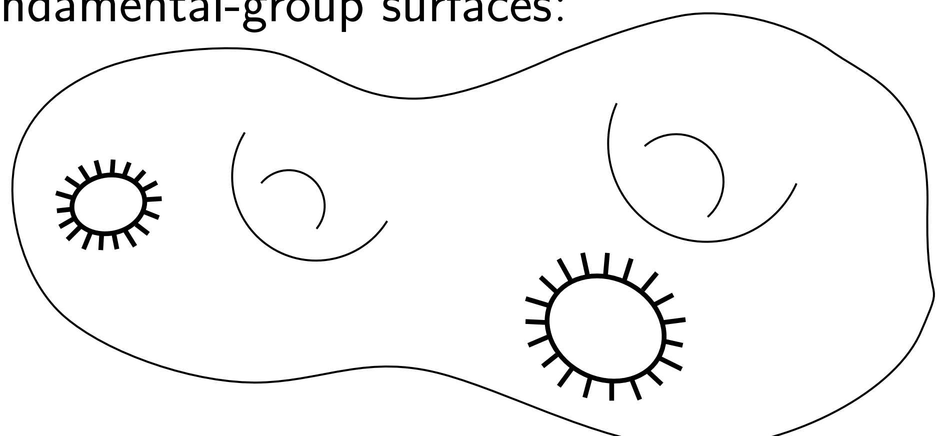
the monomer-monomer correlation $M_{b_1 \cdots b_n w_1 \cdots w_n}$ is given by

$$\frac{Z_{\Gamma_{b_1 \cdots b_n w_1 \cdots w_n}}}{Z_\Gamma}.$$

In particular, every b_ℓ adjacent to $w_\ell \Longrightarrow$ dimer $(i_{b_\ell} j_{w_\ell}) \mid i, j, \ell \in \mathbb{N}^+$:



Remark. Monomer-monomer correlations are a special case of dimer models for nontrivial fundamental-group surfaces:



Remark. $|\{[K]\}| = 2^{2g+2n-1}$, where $2n = |\text{vertices}|$.

1.7 Partition function as sum of Pfaffians

Theorem.

$$Z = \frac{1}{2^g} \sum_{[K]} \mathsf{Arf}(q^K_{D_0}) \cdot \varepsilon^K(D_0) \cdot \mathsf{Pf}(\Gamma^K) \qquad \Big| \, \mathsf{Arf}(\cdot) \in \{\pm 1\}.$$

such that:

$[K] =$ equivalence classes of Kasteleyn orientations, 2^{2g} in total

$q^K_{D_0} =$ quadratic form on $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$ corresponding to Kasteleyn orientation, for reference configuration D_0

$$\varepsilon^K(D_0) = (-1)^{\sigma} \, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{2n-1}\sigma_{2n}} \qquad \Bigg| \begin{array}{l} \sigma \in \mathsf{Aut}(\mathscr{D}) \\ \{[\sigma]\} \subseteq \mathcal{S}_{2n} / (\mathcal{S}_n \times \mathcal{S}_2^n). \end{array}$$

Proof. $\heartsuit$.

Remark. In particular:

(i). For bipartite graphs on $\overline{\mathcal{M}}_g$:

height function =

= section of the non-trivial $\mathbb{Z}$-bundle.

(ii). For fundamental cycles $(a_1, \ldots, a_g, b_1, \ldots, b_g)$:

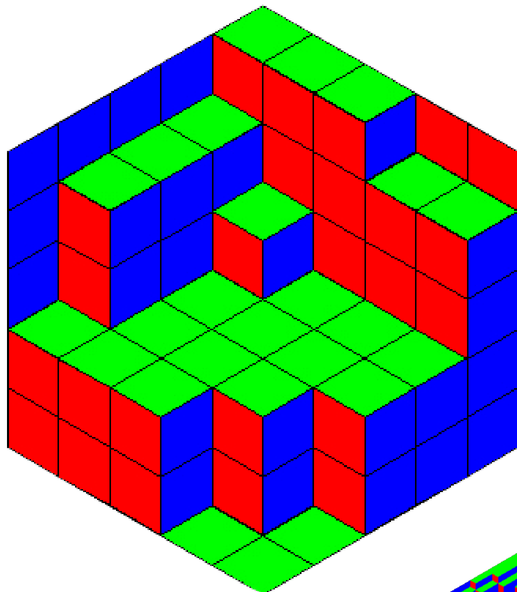
$$Z\big(\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_g}, \mathcal{H}_{b_1}, \ldots, \mathcal{H}_{b_g}\big) =$$

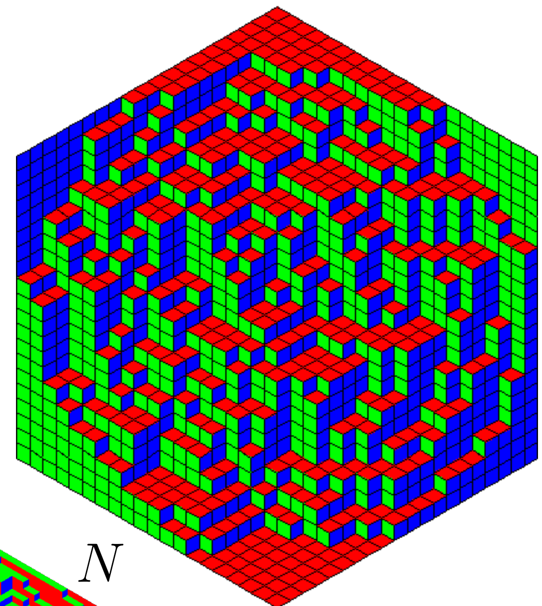
$$= \sum_D \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{i=1}^{g} \exp\Big(\sum_i \mathcal{H}_{a_i} \Delta_{a_i} h + \\ + \sum_i \mathcal{H}_{b_i} \Delta_{b_i} h \Big)$$

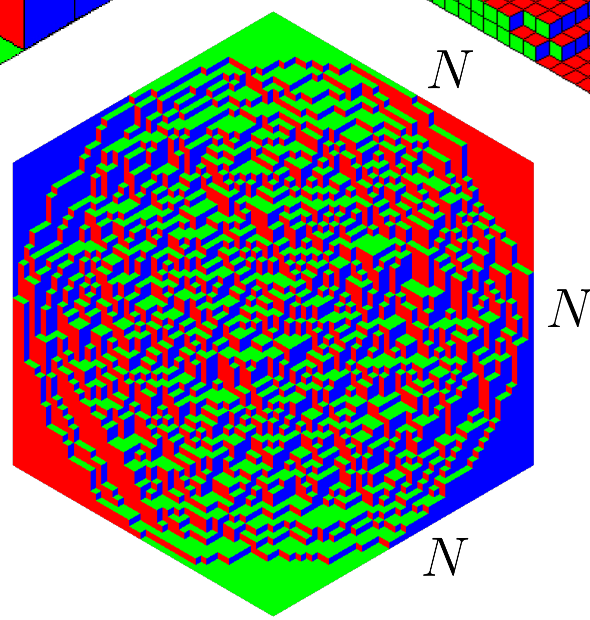
where $\Delta_C h =$

= change in height function along noncontractible cycle C on $\overline{\mathcal{M}}_g$.

1.8 Thermodynamic limit of bulk interactions





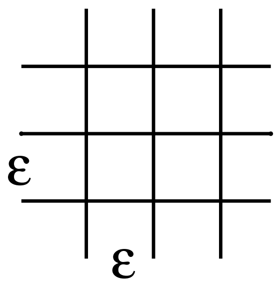


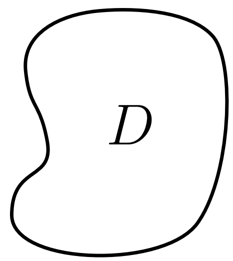
uniform measure

$$\mathrm{Prob}(h) = \frac{1}{|\mathcal{H}_\Gamma|}$$

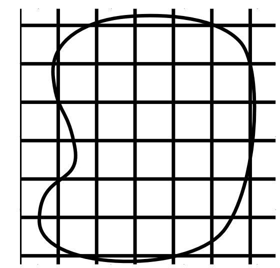
$$N \longrightarrow \infty\,.$$

Theorem (Schur process; Okounkov & R). Let $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \,|\, D \subset \mathbb{R}^2$;





such that: $\begin{cases} \varepsilon \longrightarrow 0, \ |D_\varepsilon| \longrightarrow \infty \\ D_\varepsilon = \varphi_\varepsilon(\mathbb{Z}^2) \cap D \end{cases}$



.

Then, for cube-stack with measure

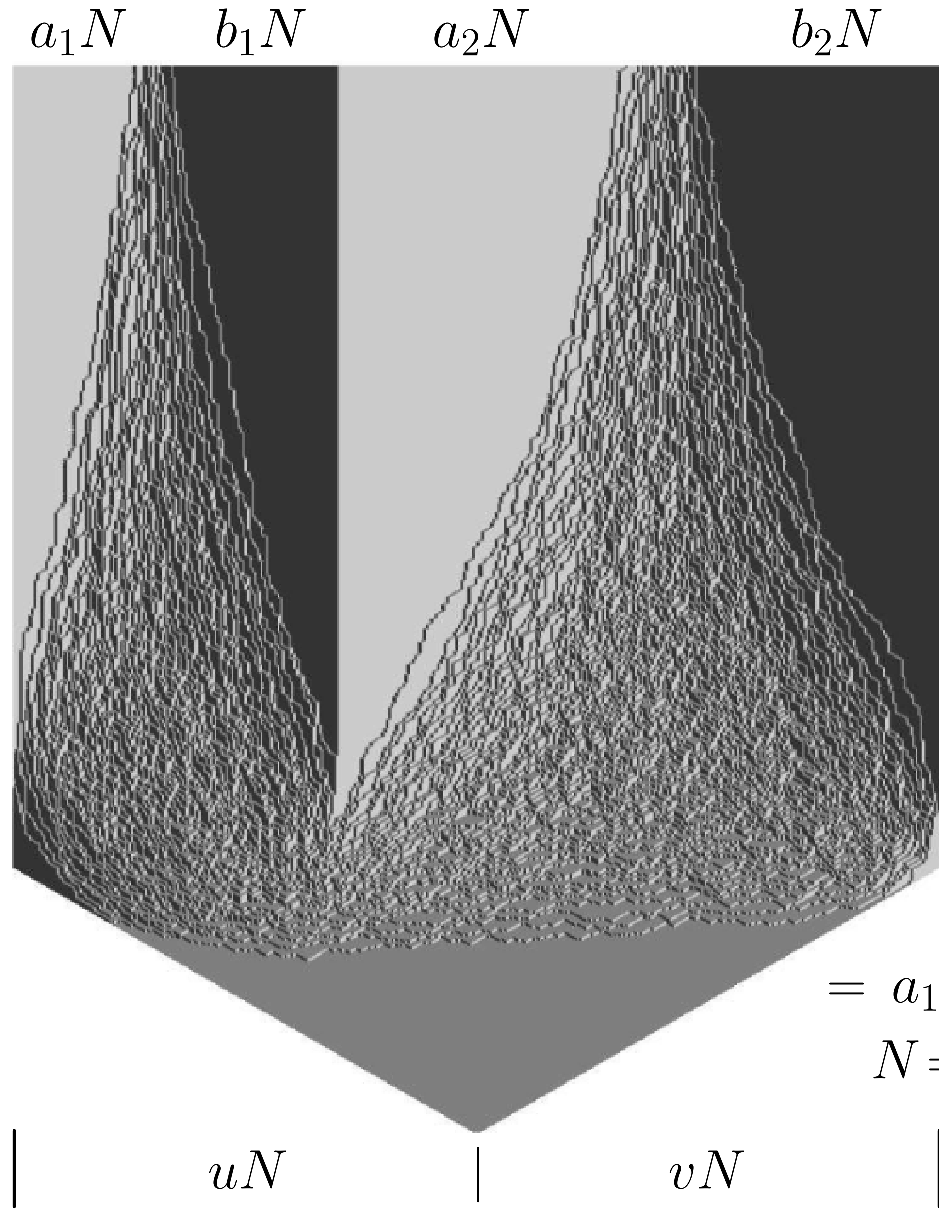
$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \quad \left| \begin{array}{l} \pi \in \mathcal{H}_\Gamma \\ \pi \cong D, \end{array} \right.$$

there is existence of:

Thermodynamic limit $(|D_\varepsilon| \longrightarrow \infty)$ +

+ *Scaling limit* $(q = e^{-\varepsilon}, \ \varepsilon \longrightarrow +0)$.

Proof. $\heartsuit$.



where $u+v=$
$= a_1+a_2+b_1+b_2;$
$N=\varepsilon^{-1},\ q=e^{-\varepsilon}.$

2 Special cases

Points:

(i) Reformulate the Grassmann integral for special genus-g domains

(ii) Find thermodynamic $\ln(\cdot)$ steepest-descent and variational-principle

(iii) State conjecture for large deviation functional, Green's function $\langle \cdots \rangle$

2.1 Grassmann integral kernel

Pairing, $\bigwedge^{\bullet} V^* \otimes \bigwedge^{\bullet} V \longrightarrow \mathbb{R}$:

$$\left\langle \varphi(a^*), \psi(a) \right\rangle \overset{\text{def}}{=\!=} \varphi_0\, \psi_0 + \sum_{k=1}^{2n} \varphi_k \psi_k + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \varphi^{\sigma(k)\cdots\sigma(1)}\, \psi^{\sigma(1)\cdots\sigma(k)} =$$

$$= |\psi_0|^2 + \sum_{k=1}^{2n} \int_{\sigma(k)<} |\psi^{\sigma(1)\cdots\sigma(k)}|^2\, d^{\,2n}a, \quad \forall\, |\psi|^2 \propto |\varphi|^2 \in \mathbb{R}$$

such that for V basis $(a_1, \dots, a_{2n})$ and V dual space V^* basis $(a_1^*, \dots, a_{2n}^*)$:

$$\bigwedge^{\bullet} V \ni \psi(a) = \psi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \psi^{\sigma(k)<}\, a_{\sigma(k)<} \left|\begin{array}{l} \bigwedge^{k} V \ni \sum \psi^{\sigma(k)<}\, a_{\sigma(k)<} \\ \sigma(k)< \;=\; (\sigma(1) < \cdots < \sigma(k)) \end{array}\right.$$

$$\bigwedge^{\bullet} V^* \ni \varphi(a^*) = \varphi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)>} \varphi^{\sigma(k)>}\, a^*_{\sigma(k)>} \left|\begin{array}{l} \bigwedge^{k} V^* \ni \sum \varphi^{\sigma(k)<}\, a^*_{\sigma(k)>} \\ \sigma(k)> \;=\; (\sigma(1) > \cdots > \sigma(k)) \end{array}\right.$$

where $\bigwedge^{\bullet} V =$ Grassmann algebra, on basis $(a_1, \dots, a_{2n}) \subseteq V$, generated by

$$\left\{\begin{array}{l} a_0 = 1;\; a_{\sigma(k)<} = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},\ \forall\, \sigma(k)< \;=\; (\sigma(1), \dots, \sigma(k)) \mid \sigma(1) < \cdots < \sigma(k); \\ a_{\sigma(\xi)} \otimes a_{\sigma(\eta)} + a_{\sigma(\eta)} \otimes a_{\sigma(\xi)} = 0, \forall\, \xi, \eta = 1, \dots, k;\; \sigma(\xi), \sigma(\eta), k = 1, \dots, 2n \end{array}\right\}.$$

Fixing integrals on $\bigwedge^\bullet V,\ \bigwedge^\bullet V^*,\ \bigwedge^\bullet(V^*\otimes V)$ by choosing

$$a_1, \ldots, a_{2n} \in \bigwedge^{2n} V, \quad a^*_{2n}, \ldots, a^*_1 \in \bigwedge^{2n} V^*$$

and

$$a^*_{2n}, \ldots, a^*_1,\ a_1, \ldots, a_{2n} \in \bigwedge^{2n} V^* \otimes \bigwedge^{2n} V$$

then

$$\int \bigotimes_{i=1}^{\eta} a^*_{\sigma(i)} \bigotimes_{i=1}^{\eta} a_{\tau(i)}\ da^*\, da\ = \begin{cases} 0 & ,\quad \eta \neq 2n \\ (-1)^{(\sigma\, +\, \tau\, +\, n(2n-1))} & ,\quad \eta = 2n \end{cases}$$

$$\sigma\ :\ (\sigma(1), \ldots, \sigma(2n)) \longrightarrow (1, \ldots, 2n)$$
$$\tau\ :\ (\tau(1), \ldots, \tau(2n)) \longrightarrow (1, \ldots, 2n)\,.$$

Lemma.

$$\langle \varphi(a^*), \psi(a) \rangle\ = \int \exp\Big(\sum_i a^*_i a_i\Big)\ \varphi(a^*)\ \psi(a)\ da^*\, da\,.$$

Proof. $\heartsuit$.

Lemma. *Let* $A\colon V \longrightarrow V$ *by*

$$\psi_A(a) = \sum_{\{i\}_<,\{j\}_<} a_{\{i\}_<} A_{\{i\}_<\{j\}_<} \psi_{\{j\}_<}$$

$$= \psi_0 \oplus A\psi_1 \oplus A^{\otimes 2}\psi_2 \oplus \cdots$$

then

$$\psi_A(b) =$$

$$= \int \exp(-a^* A\, b)\, \exp(-a^* a)\, \psi(a) da^*\, da.$$

Proof. $\heartsuit$.

Lemma.

$$\int \exp(-a^* A\, b)\, \exp(-a^* a)\, \exp(-B^* B\, a)\, da^* da =$$

$$= \exp(-b^* B\, A\, b)\,.$$

Proof. $\heartsuit$.

Remark. Therefore, $\exp(-b^* A\, b) =$ "integral kernel" of A acting on $\bigwedge^{2n} V$.

2.2 Vertex operators

(i). The Fermionic Fock space, i.e. $\langle V_m \rangle \in \mathbb{C}^{\,\mathbb{Z}+\frac{1}{2}}$ is given by

$$F = \left\{ V_{m_1} \wedge V_{m_2} \wedge \cdots \;\middle|\; \begin{array}{l} m_i \in \mathbb{Z}+\frac{1}{2} \\ m_{i+1} = m_i - 1 \\ i \gg 1 \end{array} \right\}.$$

(ii). The Clifford algebra:

$$Cl_{\mathbb{Z}} = \left\langle \psi_m \,,\, \psi_m^* \right\rangle \;\left|\; \begin{array}{l} m \in \mathbb{Z}+\frac{1}{2} \\ \psi_m \psi_{m'} + \psi_{m'} \psi_m \;=\; \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* \;=\; 0 \\ \psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m \;=\; \delta_{m\,m'} \,. \end{array} \right.$$

(iii). The Clifford algebra act on the Fock space F:

$$\psi_m \; v_{m_1} \wedge v_{m_2} \wedge \cdots \;=\; v_m \wedge v_{m_1} \wedge v_{m_2} \wedge \cdots$$

$$\psi_m^* \; v_{m_1} \wedge v_{m_2} \wedge \cdots \;=\; \sum_{i=1}^{\infty} (-1)^i \;\delta_{m_i,\,m}\; v_{m_1} \wedge \cdots \wedge \widehat{v_{m_1}} \wedge \cdots$$

(iv). The Heisenberg algebra:

$$\left\langle \alpha_n \right\rangle \;\; \left| \begin{array}{l} n \in \mathbb{Z}\backslash\{0\} \\ [\alpha_n,\, \alpha_{n'}] \;=\; -n\, \delta_{n,\,-n'}\,. \end{array} \right.$$

(v). The Heisenberg algebra act on the Fock space F:

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m \,\in\, \mathbb{Z}+\frac{1}{2}} \psi_{m+n}\, \psi_m^*\,.$$

- As operator in F:

$$\Big[\alpha_n\,,\, \psi_\xi\Big] = \psi_{\xi+n} \;\;,\;\; \Big[\alpha_n\,,\, \psi_\xi^*\Big] = -\psi_{\xi-n}^*\,.$$

(vi). The vertex operators in F:

$$\Gamma_\pm(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\, \alpha_{\pm n}\Big) \;\; \left| \begin{array}{l} \big(\Gamma_-(x)v,\, w\big) = \\ \quad = \big(v,\, \Gamma_+(x)w\big) = \\ \quad = \big(\Gamma_+(x)w,\, v\big). \end{array} \right.$$

(vii). The commutation relations:

$$\begin{aligned}
\Gamma_+(x)\,\Gamma_-(y) &= (1-x) \cdot \Gamma_-(y)\,\Gamma_+(x) \\
\Gamma_+(x)\,\psi(z) &= (1-z^{-1}\,x)^{-1} \cdot \psi(z)\,\Gamma_+(x) \\
\Gamma_-(x)\,\psi(z) &= (1-x\,z)^{-1} \cdot \psi(z)\,\Gamma_-(x) \\
\Gamma_+(x)\,\psi^*(z) &= (1-z^{-1}\,x) \cdot \psi^*(z)\,\Gamma_+(x) \\
\Gamma_-(x)\,\psi^*(z) &= (1-z\,x) \cdot \psi^*(z)\,\Gamma_-(x).
\end{aligned}$$

(viii). The eigenvectors:

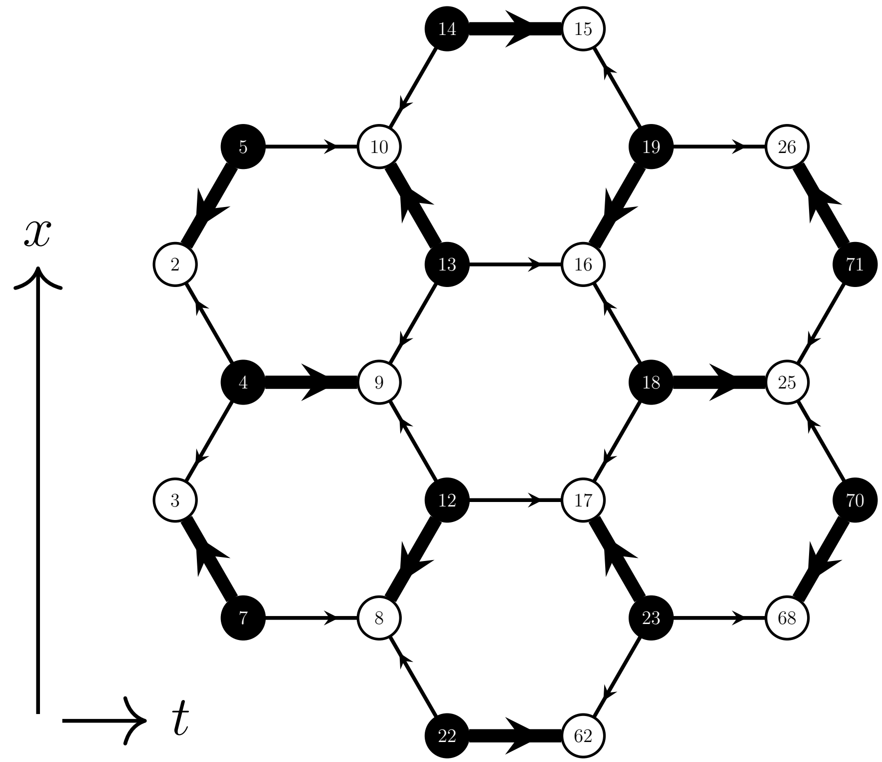
$$\Gamma_-(x) \prod_i \psi^*(w_i) \prod_j \psi^*(z_j)\, V_0^{(n)} =$$

$$= \prod_i (1-x\,z_i)^{-1} \prod_j (1-x\,w_j) \prod_i \psi^*(w_i) \prod_j \psi^*(z_j)\, V_0^{(n)}$$

where $V_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

2.3 Fermionic Kasteleyn operator

Bipartite hexagonal lattice for the one cube of "two-color" tiles is given by:



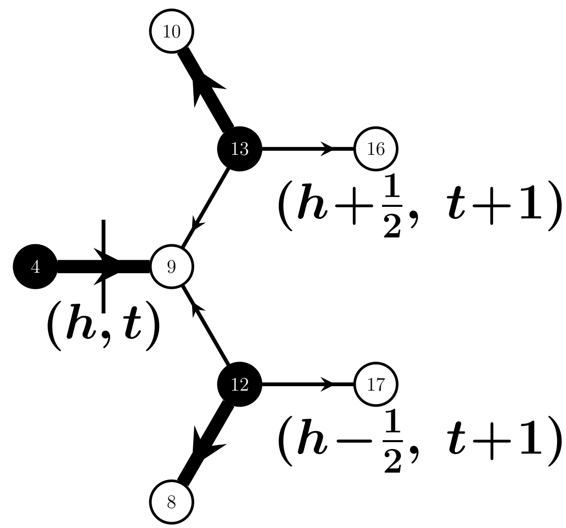
such that general parameterization for bipartite hexagonal lattice is given by:

$$b(h,t) = (h,\ t - \tfrac{1}{2})$$

$$w(h,t) = (h,\ t + \tfrac{1}{2}).$$

Kasteleyn matrix for above-chosen $b \sim w$ diagram is given by

$$K(h,t) \ = \ (h,t) \ - \ \left(h+\tfrac{1}{2},\, t+1\right) \ + \ x_{h,t}\left(h-\tfrac{1}{2},\, t+1\right).$$



Placing Fermions $a^*_{h,\,t}\,,\ a_{h,\,t}$ respectively, at $b(h,t)$ and $w(h,t)$

$$a^* K a \ = \ \sum_{h,t} a^*_{h,t}\, a_{h,t} \ - \ \sum_{h,t} a^*_{h+\frac{1}{2},\,t+1}\, a_{h,t} \ + \ \sum_{h,t} a^*_{h-\frac{1}{2},\,t+1}\, a_{h,t}\, x_{h,t} \ =$$

$$= \ \sum_{t} \Big(a^*_t a_t \ + \ a_t V a^*_{t+1} \ + \ a_t V^{-1} x_t\, a^*_{t+1} \Big).$$

Considering boundary conditions

$[Diagram]$

Prob $(\pi) \propto$

$\propto \prod_t q_t^{|\pi(t)|}$

$\begin{pmatrix} \text{in previous} \\ \text{notations} \\ q_{h,t} = q_t \end{pmatrix}$

assuming $x_{h,t} = x_t$.

Theorem.

$$Z = \int \exp\big(a^* A\, a\big)\, da^*\, da \;=$$

$$= \Big\langle \Gamma_-\big(x_{-\frac12}\big)\cdots\Gamma_-\big(x_{u_0+\frac12}\big)\,\Gamma_+\big(x_{\frac12}\big)\cdots\Gamma_+\big(x_{u_1+\frac12}\big)\,V_0^{(0)}\,,\quad V_0^{(0)}\Big\rangle\,.$$

Proof. Outline:

$$\int \cdots\ \exp\big(a^*_{t-1}\,a_{t-1}\big)\cdot \exp\big(a_{t-1}\,\big(V - V^{-1}\Gamma_t\big)\,a^*_t\big)\cdot$$
$$\cdot\ \exp\big(a^*_t\,a_t\big)\cdot\ \exp\big(a_t\,\big(V - V^{-1}\Gamma_t\big)\,a^*_{t+1}\big)\cdots \quad =$$

$$= \ \cdots\ \underbrace{\widetilde{\big(V - V^{-1}\Gamma_{t-1}\big)}^{\,-1}}_{\Gamma_+(x_t)}\cdot\underbrace{\widetilde{\big(V - V^{-1}\Gamma_t\big)}^{\,-1}}_{\Gamma_-(x_t)}\cdots$$

where $\Gamma_+(x_t)$ and $\Gamma_-(x_t)$ each depends on t such that

$$\widetilde{A} = A,\ \text{ where } V \hookleftarrow \text{ is lifted to } \textstyle\bigwedge^{\frac{\infty}{2}} V \ \Big|\, V = \bigoplus_{m\in\mathbb{Z}+\frac12} \mathbb{C}\, v_h$$

with boundary conditions, etc. □

Remark. Direct proof exists combinatorially without Kasteleyn method.

Corollary.

$$Z = \prod_{m=\frac{1}{2}}^{u_1-\frac{1}{2}} \prod_{m'=u_0+\frac{1}{2}}^{-\frac{1}{2}} \left(1 - x^-_{m'} x^+_m\right)^{-1}.$$

Theorem. (Okounkov & R., 2005).

$$\left\langle \sigma_{(h_1 t_1)} \cdots \sigma_{(h_k t_k)} \right\rangle = \det(K((t_i,h_i),(t_j,h_j)))_{1 \leqslant i,j \leqslant k}$$

$$K((t_i,h_i),(t_j,h_j)) =$$
$$= \frac{1}{(2\pi i)^2} \int\limits_{|z|<R(t_1)} \int\limits_{|z|<\widetilde{R}(t_2)} \frac{\Phi_-(z,t_1) \;\; \Phi_+(w,t_2)}{\Phi_+(z,t_1) \;\; \Phi_-(w,t_2)} \cdot$$
$$\cdot \frac{1}{z-w} \cdot z^{\left(-h_1 - B(t_1) - \frac{1}{2}\right)} \cdot w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \, dz \, dw$$

such that:

$|w|<|z|,\ t_1 \geqslant t_2$

$|w|>|z|,\ t_1 < t_2$

$$R(t) = \min_{m>t}((x^+_m)^{-1}),\ \widetilde{R}(t) = \max_{m<t}(x^-_m),\ B(t) = \frac{|t|}{2} - \frac{|t-u_0|}{2}$$

$$\Phi_+(z,t) = \prod_{m>\max(t,\frac{1}{2})} (1 - z\,x^+_m),\quad \Phi_-(z,t) = \prod_{m<\max(t,-\frac{1}{2})} (1 - z^{-1}\,x^-_m).$$

2.4 Thermodynamic scaling limit

$[Diagram]$

$$\left.\begin{aligned} x_m^+ &= aq^m \\ x_m^- &= a^{-1}q^m \end{aligned}\right\} \text{assumed}$$

corresponding to $\text{Prob}(\pi) \propto q^{|\pi|}$.

2.5 Kasteleyn operator asymptotics

Consider limit $\varepsilon \longrightarrow 0 \mid q = e^{-\varepsilon},\ u_1 = \varepsilon^{-1} v_1,\ u_0 = \varepsilon^{-1} v_0$ for fixed v_1, v_0:

$$Z = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - x_m^- x_n^+)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - q^{m-n})^{-1}$$

$$\langle |\pi| \rangle = q \frac{\partial}{\partial q} \ln Z$$

$$\ln Z = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln(\underbrace{1 - e^{-s+t}}_{\text{2D partition function}})\, ds\, dt + \cdots$$

$$\langle |\pi| \rangle = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1-e^{t-s}}}_{\text{3D volume function}}\, ds\, dt + \cdots$$

Consider limit $\varepsilon \longrightarrow 0$ where $t_i = \varepsilon^{-1}\tau_i,\ h_1 = \varepsilon^{-1}\chi_i,$ for fixed $\tau_i,\ \chi_i:$

$[Diagram]$ $(\tau_i,\ \chi_i)$ *in the bulk*

$$K((t_1,h_1),(t_2,h_2)) =$$

$$= \frac{1}{(2\pi i)^2} \int_{C_z} \int_{C_w} \exp\big(\varepsilon^{-1}(S(z,t_1,\chi_1) - S(z,t_2,\chi_2))\big) \cdot \cdot (zw)^{1/2} (z-w)^{-1}\, dz\, dw$$

where

$$S(z,t,\chi) =$$

$$= -(\chi + \frac{\tau}{2} - u_0) \ln Z + + \mathrm{Li}_2(ze^{-v_0}) + \mathrm{Li}_2(ze^{-v_1}) - \mathrm{Li}_2(z) - \mathrm{Li}_2(ze^{-\tau})$$

and

$$\mathrm{Li}_2(z) = \int_0^z t^{-1} \ln(1-t)\, dt.$$

2.6 Critical points

$$\exp\left(\chi+\frac{\tau}{2}\right) = \frac{(1-ze^{-v_0})(1-ze^{-v_1})}{(1-z)(1-ze^{-\tau})}$$

gives quadratic equation, implying a discriminant for two real solutions or two complex-conjugate solutions, or a zero-discriminant.

$[Diagram]$

$$\boxed{\partial_\chi h_0\,(\tau,\chi) = \frac{1}{\pi}\,\mathsf{arg}(z_0)}$$

$$\left\langle \sigma_{(h,t)}\right\rangle = K((t,h),(t,h)) \longrightarrow \varepsilon\,\partial_\chi h_0(\tau,\chi)$$

2.7 Steepest descent

$$K((t_1,h_1),(t_2,h_2)) \;=\; -\frac{\varepsilon}{2\pi}\cdot\Big(\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\}}{(z_1-w_2)\,\sqrt{-w_2S_2''(w_2)}\;\sqrt{z_1S_1''(z_1)}}\;-$$

$$-\;\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\}}{(z_1-\overline{w}_2)\,\sqrt{-\overline{w}_2S_2''(\overline{w}_2)}\;\sqrt{z_1S_1''(z_1)}}\;+\;c.\,c.\Big)\cdot(1+O(1))$$

That is, for $\mathcal{H}_+ = \{z\in\mathbb{C},\ \mathrm{Im}\, z>0\} \mid z_0(\chi,\tau) =$ inner process, such that

$$z_1 \;=\; z_0(\chi_1,\tau_1)$$

$$w_2 \;=\; z_0(\chi,\tau),$$

$$K((t_1,h_1),(t_2,h_2)) \;=$$

$$=\;\frac{\varepsilon}{2\pi}\exp\{\varepsilon^{-1}(\mathrm{Re}(S(z_0(\chi_1,\tau_1)))-\mathrm{Re}(S(z_0(\chi_2,\tau_2))))\}\,\cdot$$

$$\cdot\Big(\frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(w_2)))\}}{(z_1-w_2)}\;+$$

$$+\;\frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(\overline{w}_2)))\}}{(z_1-\overline{w}_2)}\;+\;c.\,c.\Big)\cdot(1+O(1))\qquad(*).$$

Hence, solution for Kasteleyn-Fermions to free Dirac-Fermions convergence:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + $$
$$+\ \psi_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+O(1))$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^*_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi^*_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + $$
$$+\ \psi^*_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+O(1))$$

such that

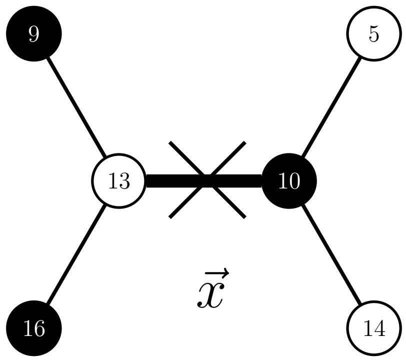
$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\pm(w)) = \frac{1}{z-w}$$

$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\mp(w)) = \mathbb{E}(\psi^*\,\psi^*) = \mathbb{E}(\psi\,\psi) = 0$$

where $\psi^*_\pm(z)$, $\psi_\pm(w)$ are spinors:

$$\psi^*_\pm(z) = \psi^*_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\ ,\quad \psi_\pm(z) = \psi_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\ .$$

Remark. The correlation is given by:



$$\left\langle (\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle)\, (\sigma_{\vec{x}_2} - \langle \sigma_{\vec{x}_2} \rangle) \right\rangle = K_{12} K_{21} =$$

$$= \frac{\varepsilon^2}{(2\pi)^2} \left(\frac{\frac{\partial z_1}{\partial x_1} \frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} - \frac{\frac{\partial z_1}{\partial x_1} \frac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} + c.\,c. \right) \times$$

$$\times\ (1 + O(1)).$$

In particular,

$$\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle = \varepsilon\, \partial_x\, \varphi(z_0(\tau, x)) + \cdots$$

$\varphi(z)$ = Gaussian free field on $\mathcal{H}_+$

such that the Green's function of Dirichlet problem on $\mathcal{H}_+$ is given by

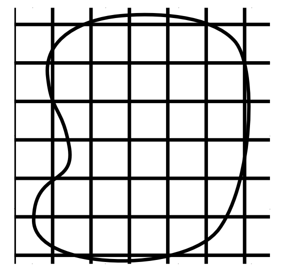
$$\langle \varphi(z)\, \varphi(w) \rangle = \frac{1}{2\pi} \ln \left| \frac{z - w}{z - \overline{w}} \right|$$

and, the Bose-Fermi correspondence is given by

$$\partial_x \varphi = :\widetilde{\psi}(z, \overline{z})\ \widetilde{\psi}(z, \overline{z}): \ \cdots\ .$$

2.8 Kasteleyn operator scaling limit

Let $\Gamma = D_\varepsilon = \varphi_\varepsilon(L) \cap D,$ for an arbitrary lattice L:



Then

$$(\Gamma_\Gamma^K)_x \cdot G_{x,y} \;=\; \delta_{x,y}$$

for $\Gamma_\Gamma^K =$ difference operator, where $\varepsilon \longrightarrow 0$ asymptotically for $G_{x,y}$.

Particular cases.

(i). Hexagonal lattices: Consider the weighted as above, such that

$$q_t = e^{-\varepsilon\, f(t)},\; t = \frac{\tau}{\varepsilon},\; \varepsilon \to 0.$$

Theorem.

$$G_{x,y} = \textit{same as } (*),\ \textit{with different } z_0(\tau, x).$$

(ii). Periodic lattices: Consider variational principle.

2.9 Variational principle

(i). For the $N \times M$ torus

$$[Diagram]$$

$$Z(H,V) = \sum_D \prod_\ell \omega(\ell) \exp(H\Delta_a h_D + V\Delta_b h_D)$$
$$= \frac{1}{2}\Big(\mathsf{Pf}(A^{K_1}) + \mathsf{Pf}(A^{K_2}) + \mathsf{Pf}(A^{K_3}) - \mathsf{Pf}(A^{K_4})\Big)$$

where $N, M \to \infty$, $\frac{N}{M} = \text{fixed}$.

And, $\omega(\ell) = 1$ gives Kasteleyn matrices' eigenvalues by Fourier transform.

Theorem. (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$\lim_{N,M \longrightarrow \infty} \frac{1}{NM} \ln Z_{NM} = \oint \oint \ln|1+zw| \frac{dz}{z}\frac{dw}{w}$$
$$= f(H,V) = \begin{cases} |z| = e^H \\ |w| = e^V. \end{cases}$$

(ii). Taking Legendre transform

$$\boldsymbol{\sigma}(s, t) \;=\; \max_{H,V}\,(H_s + V_t - f(H, V))$$

then

$$\sum_D 1 \;=\; \sum_D \prod_D w\,(e) \;=\; \exp(MN\,\boldsymbol{\sigma}(s, t)\cdot(1+O(1)))$$

where

$$\frac{\Delta_a h_D}{M} = s, \quad \frac{\Delta_b h_D}{N} = t, \quad M, N \to \infty, \quad \frac{N}{M} \text{ fixed.}$$

(iii). For domain

$$[Diagram]$$

$\Delta_a h = sM,\ \Delta_b h = tN.$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\sum_D 1 \;=\; \exp(MN\,\boldsymbol{\sigma}(s, t)\cdot(1+O(1)))$$

with the boundary conditions of height function h_D.

(iv). For domain

$$[Diagram] \qquad M_i \times N_j$$

$$
\begin{aligned}
Z_{D_{\boldsymbol{\varepsilon}}} &= \sum_{\left\{\substack{\text{values of}\\ \text{height functions}\\ \text{on}\\ \text{boundaries}\\ \text{between rectangles}}\right\}} Z_{\underset{M_i}{\square} N_j}(h_{\text{bound}}) \\
&= \sum_{\{\Delta_x h,\ \Delta_y h\}_{ij}} \exp\left(\sum_{\underset{M_i}{\square} N_j} M_i\, M_j\ \boldsymbol{\sigma}\Big(\frac{\Delta_x h}{M_i},\ \frac{\Delta_y h}{N_j}\Big) \right) \\
&= \exp\left(\boldsymbol{\varepsilon}^{-2} \int_D \boldsymbol{\sigma}(\partial_x h_0,\ \partial_y h_0)\ dx\, dy\ (1 + O(1))\right)
\end{aligned}
$$

where $h_0 =$ minimizer for

$$S[h] = \int_D \boldsymbol{\sigma}(\partial_x h_0,\ \partial_y h_0)\ dx\, dy.$$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln Z_{D_{\boldsymbol{\varepsilon}}} = \int_D \sigma(\overrightarrow{\nabla} h_0)\, dx\, dy$$

such that

- $h_0 =$ *minimizer*
- $0 < \partial_x h, \partial_y h < 1$
- $h_0\big|_{\partial D} = b,$ *the boundary condition appearing in the limit* $\varepsilon \longrightarrow 0.$

$$[Diagram]$$

for height function

$$\begin{aligned} h &= \varepsilon^{-1} h_0 + \varphi \\ &= \varepsilon^{-1} (h_0 + \varepsilon\varphi) \end{aligned}$$

where $h_0 =$ *limit topology*
and, $\varphi =$ *fluctuation (factor).*

2.10 Physics way of describing the fluctuations

$$S[h_0 + \varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2} \iint_D a^{ij}(x) \partial_i \varphi \, \partial_j \varphi \, d^2 x$$

$$a^{ij}(x) = \partial_i \partial_j \, \varphi(s,t) \begin{cases} s = \partial_1 h_0 \\ t = \partial_2 h_0 \end{cases}$$

such that:

- Partition function equals

$$Z = \exp(\varepsilon^{-2} S(h_0)) \int \exp\left(\frac{1}{2} \iint_D a^{ij}(x) \partial_i \varphi \, \partial_j \varphi \, d^2 x\right) D\varphi$$

where $D =$ scalar field with Riemannian metric induced by h_0;

- Correlation equals

$$\langle \varphi(x) \, \varphi(y) \rangle = G(x,y)$$

where $G =$ Green's function for $\Delta = \partial_i (a^{ij} \partial_j)$.

Conjecture. $G =$ *same as obtained by asymptotics of Kasteleyn operators.*

Remark. The conjecture = theorem, always a.s., i.e. in certain cases.

Conclusion - the limit-topology phenomena, ongoing

1. How to make such pictures of (i.e. simulate) random configuration:
   (i). Monte Carlo for $\exp(\propto 1000^2)$
   (ii). Sampling around most probable region
   (iii). Markov Chain Monte Carlo method
2. How to describe the limit topologies and fluctuations analytically:
   (i). Kasteleyn vertex partition and correlation representation
   (ii). Variational principle: Minimizing large deviation functional
   (iii). Boundary conditions

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Thank you!