Continuum Branching Process in Higher Genus: Topological Q-algebra

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Abstract

For all fixed sufficient-large genus $g \geqslant 0$, spanning dual trees of bipartite isospectral manifold superalgebra, we prove: Space $L^q(\gamma_n, E)$, free Dirac Fermion convergence $\Psi = f \cdot (1 + \mathcal{O}(1))$ in $\mathcal{O}(n^3)$ graded kernel asymptotics, discriminant steepest descent of thermodynamic limit scaling. We conjecture: Green's function $\mathscr{G}$ of large deviation Dirichlet problem for variational-principle minimizer equals correlation in the graded kernel asymptotics.

Keywords: Higher-genus, Branching-process, Topological-q-algebra

1 Characterizations

An $\mathbb{R}^n$-valued random variable $X = (X_1, \ldots, X_n)$ is Gaussian if, $\forall\, t \in \mathbb{R}^n$, $t'X = \sum_k t_k X_k$ is $\mathbb{R}$-valued Gaussian i.e. $X = \mu$ a.s. for vector μ, matrix Σ

$$\text{by} \quad \mathbb{E}[itX] = \exp\left(it'\mu - \frac{t'\Sigma t}{2} \right)$$

$$\iff \mathbb{P}\{X \in dx\} = \frac{1}{(2\pi \det \Sigma)^{n/2}} \exp\left(\frac{-(x-\mu)^T \Sigma^{-1} (x-\mu)}{2} \right) dx$$

$$\text{where} \ \ \mu_k = \mathbb{E}[X_k], \ \ \Sigma_{kl} = \text{cov}[X_k, X_l]$$

and, $\mathbb{R}$-valued random variable X is Gaussian if $X = \mu$ a.s. for constant μ

$$\text{by} \quad \mathbb{E}[itX] = \exp\left(it\mu - \frac{\sigma^2 t^2}{2} \right)$$

$$\iff \mathbb{P}\{X \in dx\} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx$$

$$\text{where} \ \ \mu = \mathbb{E}[X], \ \ \sigma^2 = \text{var}[X]$$

i.e. $\mathbb{R}^n$-valued X distribution is absolutely continuous iff Σ is non-singular.

Let X be $\mathbb{R}^n$-G, where X_i are *independent* $\Longleftrightarrow \Sigma_{i,j\neq i} = \text{cov}(X_i, X_{j\neq i}) = 0$, and X is *centered* $\Longleftrightarrow \mu = \overline{0} \Longleftrightarrow X_i$ are centered ($\mu_i = 0$).

If X is *standard* (i.e. centered and $\Sigma = I$), then
$\Longrightarrow X/(\|X\|_{L^2})$ is uniformly distributed on $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$
$\Longrightarrow UX \overset{\text{d}}{=} X$ for all orthogonal U (i.e. $U \mid U^T U = UU^T = I$).

Derivation. X is standard $\mathbb{R}^n$-G $\Longleftrightarrow X_i$ are independent standard $\mathbb{R}$-G:

$$\mathbb{E}[itX] = \prod_{i=1}^{n} \Phi(t_i) = \Phi\left(\left(\sum_{i=1}^{n} t_i^2\right)^{1/2}\right) \quad \Bigg| \; \Phi(t) = e^{-\frac{1}{2}t^2}.$$

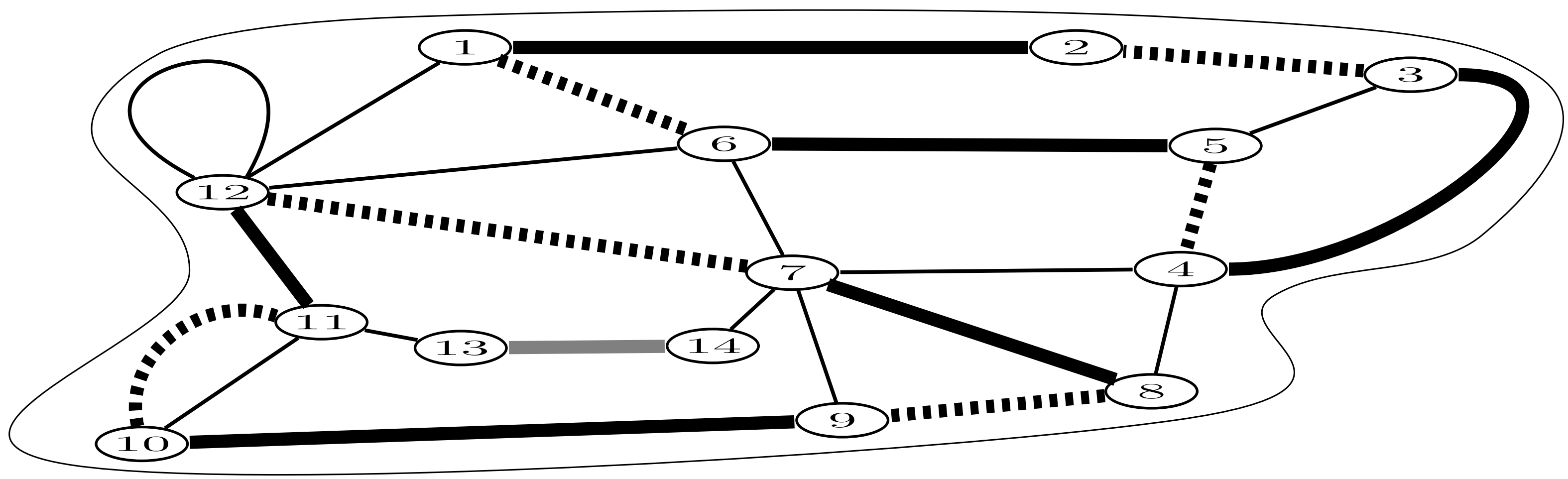
Derivation (Maxwell). $\mathbb{E}[itX] = \Phi(\sqrt{t_1^2 + t_2^2 + t_3^2}) \mid \Phi(t) = e^{-\frac{1}{2}\sigma^2 t^2}$, $\sigma^2 \geqslant 0$ for centered, random particle velocity $X = (X_1, X_2, X_3)$ ideal gas distribution.

Remark. Taking $n \longrightarrow +\infty$, for a.s. equipartition continuous density $f(x)$:

$$-\frac{1}{n} \log f^{\otimes n}(X_1, \ldots, X_n)$$
$$\longrightarrow \mathbb{E}\left[-\log f(X)\right] = -\int_S f(x) \log f(x) = \frac{1}{2} \log(2\pi e \sigma^2).$$

1.1 Partition function in height function equivalence

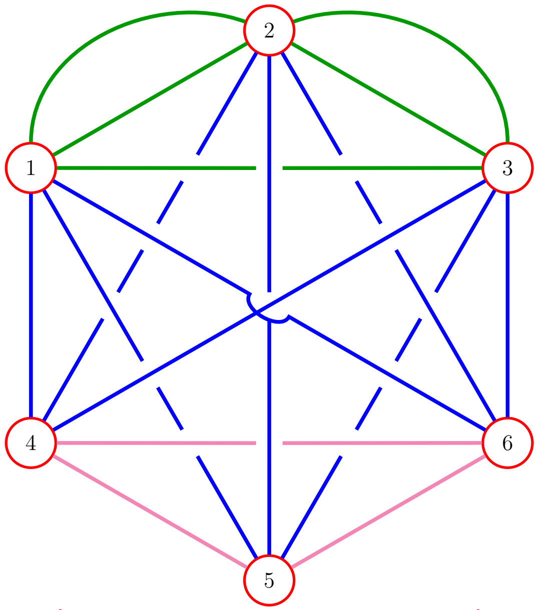
$\forall\, g \gg,$ an orientable embedding $X \subset \overline{\mathcal{M}}_g \mid \mathscr{V}_X = (i_\ell;\, i_\ell = i_\xi \neq j_\ell,\, \forall\, i \neq j)$ is partition $\sigma \in \mathsf{Aut}(\mathscr{D}) \Longleftrightarrow$ perfect-matching $D \Longleftrightarrow |\bigcap_{i_\ell j_\ell \in D}(i_\ell \in \mathscr{V}_X)| = 1$ and $\partial D = \mathscr{V}_X;$ where $\mathscr{D} = (D, \forall\, \ell);\ \overline{\mathcal{M}}_g$ orientable compact, X connected.



That is, by $\sigma_D(i_\ell j_\ell) = 1, 0,$ if $i_\ell j_\ell \in D$, resp. if $i_\ell j_\ell \notin D$,

$$\sum_{\ell \neq \xi} \sigma_{i_\ell j_\ell}(D) = \frac{1}{2}\big|\partial D = \mathscr{V}_X\big| = \frac{|\mathsf{Aut}(\mathscr{D})| \cdot |\sigma\,\big|_{\mathsf{Aut}(D)}|^{-1}}{\exp\{n \ln 2 + \sum_{k=2}^{n-1} \ln k\}}\,;\ \bigcap_{\ell \neq \xi \in D} (i_\ell, j_\xi) = \emptyset$$

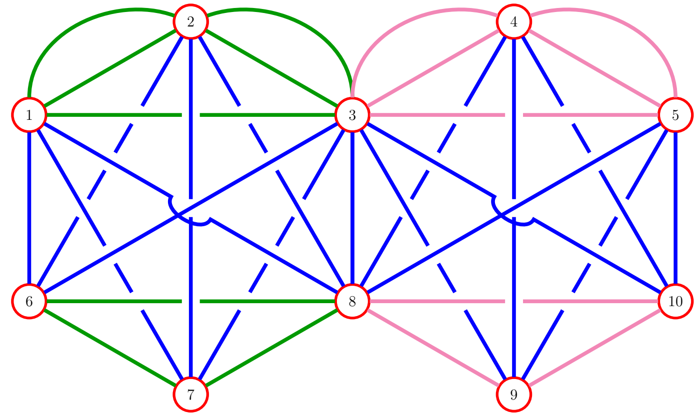
for all $\mathscr{V}_X = (i_\ell \mid i = 1, \ldots, 2n;\ \forall\, n \leqslant |\ell| < \infty,\ \ell \in \mathbb{N}^+)$, where $X \subset \overline{\mathcal{M}}_g$ is CW cell-complex i.e. face $\approx$ topological disk i.e. no hole.



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\mathbf{0}$ = non-adjoined (i, j)

$\mathbf{1}, \mathbf{1}, \mathbf{1}$ = adjoined (i, j).



$$\left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

Moreover, by $\mathbb{E}[\sigma_D(i_\ell j_\ell)\,\sigma_D(i_\xi j_\xi)] = \mathbb{E}[\sigma_D(i_\ell j_\ell)] \iff \ell = \xi$, resp. $0 \iff \sigma_D(i_\ell j_\ell) = 0$ or $(i_\ell j_\ell), (i_\xi j_\xi) \mid \ell \neq \xi$ share vertex: The local observable i.e. dimer-dimer correlation (conditional probability)

$$\left\langle \prod_{\ell=1}^{k} \sigma_D(i_\ell j_\ell) \right\rangle \overset{\text{def}}{=\!=} \text{Prob}(i_1 j_1 \in D, \ldots, i_k j_k \in D) = \mathbb{E}\left[\prod_{\ell=1}^{k} \sigma_D(i_\ell j_\ell) \right]$$

equals

$$\sum_{D \in \mathscr{D}} \prod_{\ell=1}^{k} \sigma_D(i_\ell j_\ell) \times \text{Prob}(D) = \frac{\sum_{D \in \mathscr{D}} \prod_{\ell=1}^{k} \sigma_D(i_\ell j_\ell) \prod_{\ell \in D} \omega_\ell}{\sum_{D \in \mathscr{D}} \prod_{\ell \in D} \omega_\ell}$$

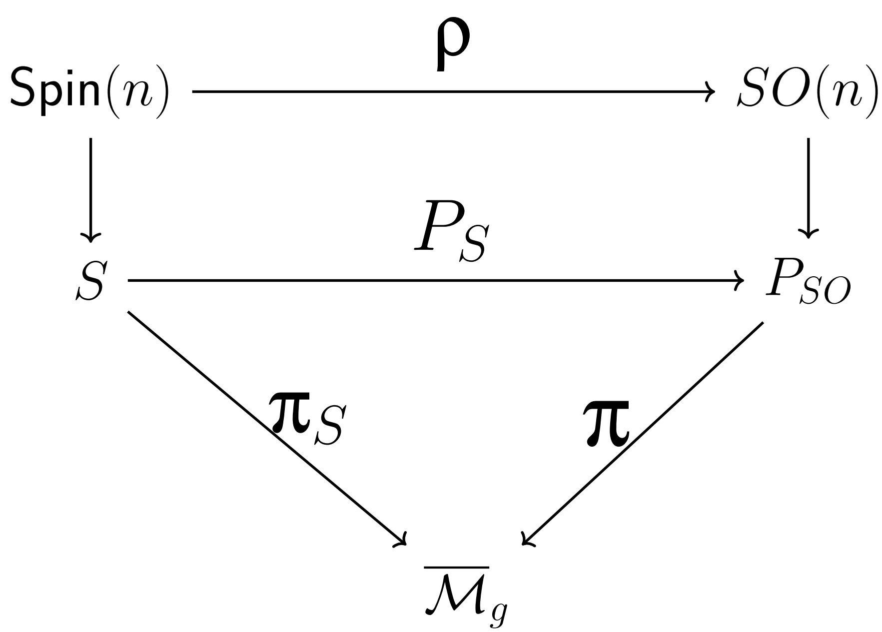
$$\neq 0 \iff \frac{1}{\boldsymbol{Z}} \sum_{D \ni (i_1 j_1), \ldots, (i_k j_k)} \omega_D \qquad \Bigg| \quad \omega_{(\cdot)} = \prod_{\ell \in (\cdot)} e^{-\frac{\Xi_{(\cdot)}}{\mathcal{K} T}}, \quad \Xi_{(\cdot)} = \sum_{\ell \in (\cdot)} \Xi_\ell$$

$$= \text{Prob}(D) \iff \left\{ D \Big\backslash \bigcap_{i_\ell j_\ell} (D \in \mathscr{D}) \right\} = \emptyset \qquad \Bigg| \quad \boldsymbol{Z} \overset{\text{def}}{=\!=} \sum_{D \in \mathscr{D}} \prod_{\ell \in D} \omega_\ell$$

for strict-sense positive partition function $\boldsymbol{Z}$ on Boltzmann weights $\omega_{(\cdot)}$ by

$$\Xi : \mathscr{E}_X \longrightarrow \mathbb{R}^+ \mid \ell \longmapsto \Xi_\ell \,.$$

Spin structure $S(\overline{\mathcal{M}}_g)$, i.e. spin (spinors) bundle $\pi_S : S \longrightarrow \overline{\mathcal{M}}_g \equiv$ Complex vector bundle on orientable Riemannian manifold $\overline{\mathcal{M}}_g$, is equivariant 2-fold cover for oriented principal orthonormal frame bundle $\pi : P_{SO} \longrightarrow \overline{\mathcal{M}}_g$, orthogonal group $SO(n)$ double-cover (structure-group) $\mathsf{Spin}(n)$, spinor space Δ_n, $\rho : \mathsf{Spin}(n) \longrightarrow SO(n)$; $\overline{\mathcal{M}}_g =$ orientable surface, is well-defined by a commutative diagram on the objects:



$$\pi_S = \pi \circ P_S$$

$$P_S\big(p, q\big) = P_S\big(p\big)\, \rho\big(q\big)$$

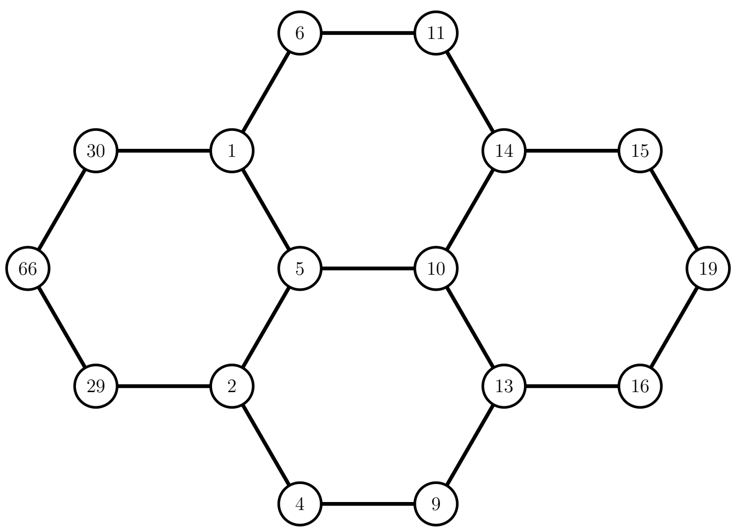
$p \in S$

$q \in \mathsf{Spin}(n)$.

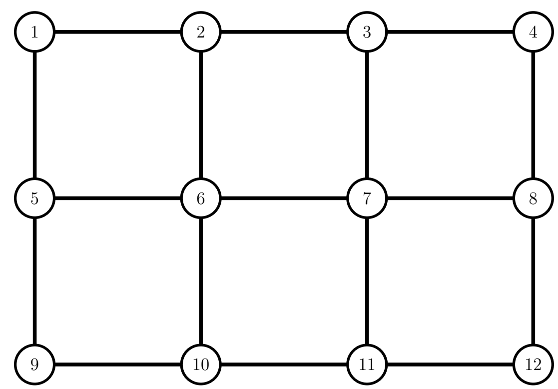
Remark. $S(\overline{\mathcal{M}}_g) \cong \sqrt{(\cdot)}$ on tangent bundle, in general for $\dim(\overline{\mathcal{M}}_g) \not> 2$.

That is, in spin structure

for the objects:



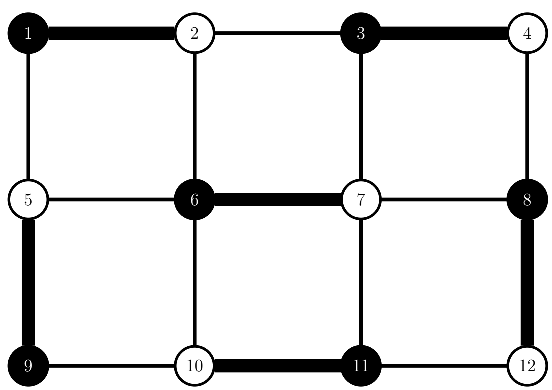
- (regular) Hexagonal grid domains.

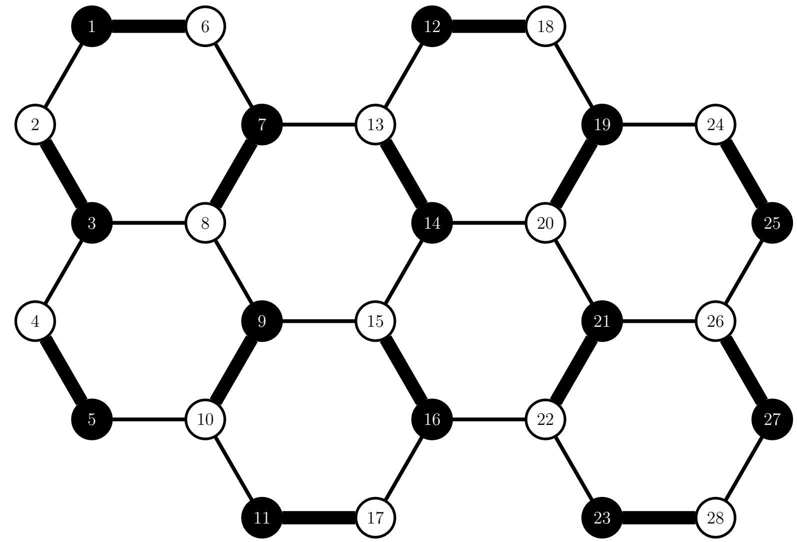


- Square grid domains.

Bipartite implies no adjacent-black (-white) vertices for all $V_X = V_X^\bullet \sqcup V_X^\circ$:
$M \times N$ vertices, $(M-1) \times (N-1)$ edges, $2n = MN$, path cartesian product.

Instance.





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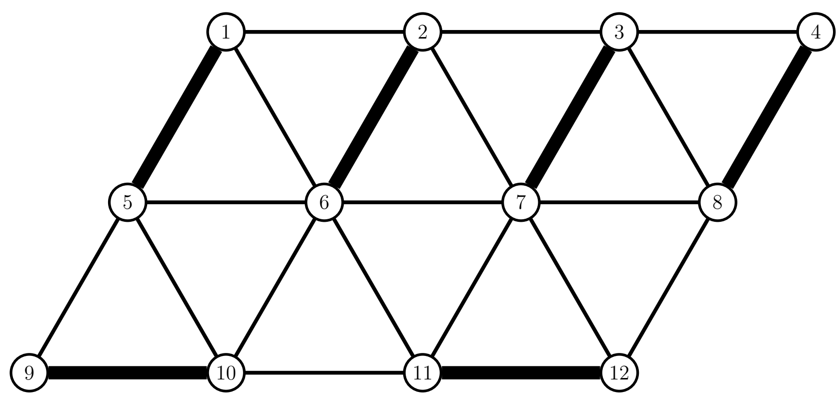
Non-instance.



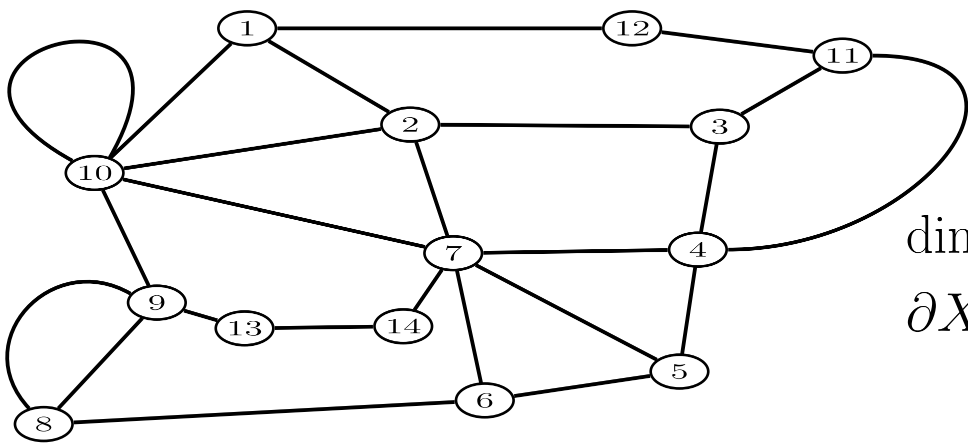
$\left(\begin{array}{l}\textit{no bipartite structure}\\ \textit{for triangular grids}\end{array}\right)$.

Proposition (combinatorial equivalence). *Given a space of dimers (of tilings resp.), there exits one-to-one combinatorial correspondence:*

$$\textbf{\textit{family (Dimers)}} \longleftrightarrow \textbf{\textit{family (Tilings)}}.$$

Proof. Let $X \subset \mathbb{R}^2$ be planar (no intersected edge) orientable. Then:

(i) 2D complex X (the union of all spanning trees T); 0-cells, 1-cells, 2-cells $=$ vertices, edges, faces, resp.



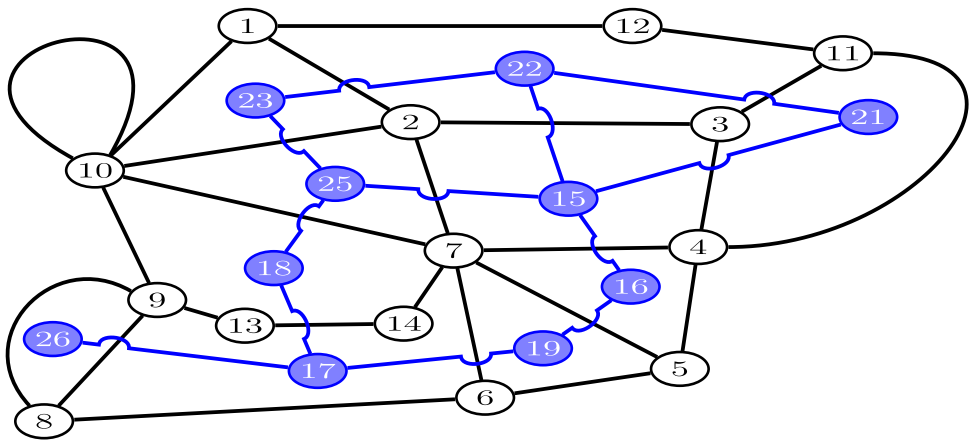
Disjoint interiors.

$\dim(\partial X^{(k)}) = (k-1) \bmod 2$

$\partial X^{(k)} =$ *boundary of two k-cells,*
$k = 0, 1, 2.$

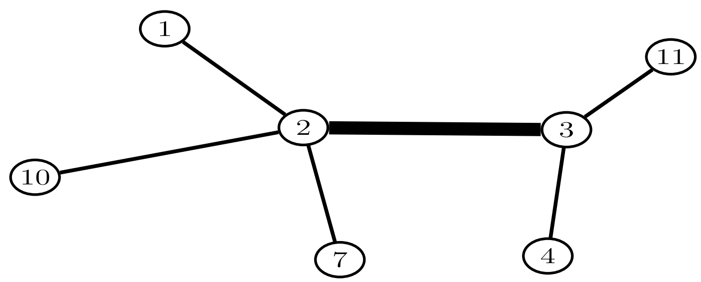
Remark. $X \subset \overline{\mathcal{M}}_g$ is generally, 1-skeleton CW-complex (resp. orientable, compact genus g cell-decomposition).

(ii) 2D dual cell complex X^* (the union of all spanning dual trees T^*); 0-cells, 1-cells, 2-cells $=$ resp. "*centers*" of 2-cells, 1-cells, 0-cells of X:

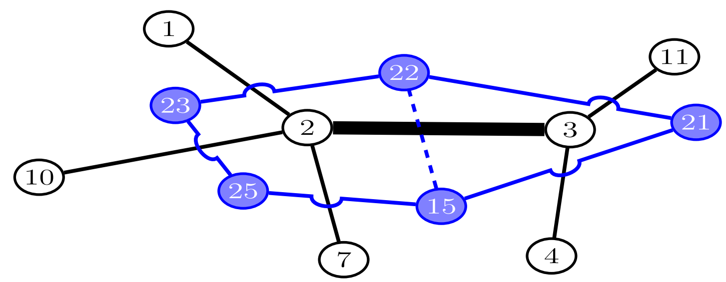


$X^* =$ dual cell complex to X.

(iii) For a dimer on X:



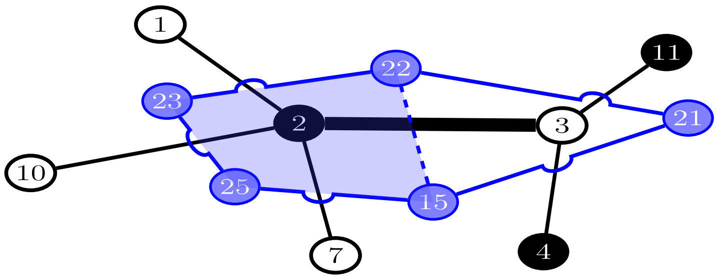
Unique pair of 2-cells on X^* share:

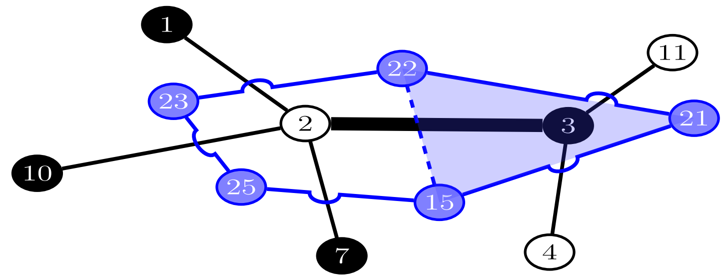


(iv) Therefore, the global bijection:

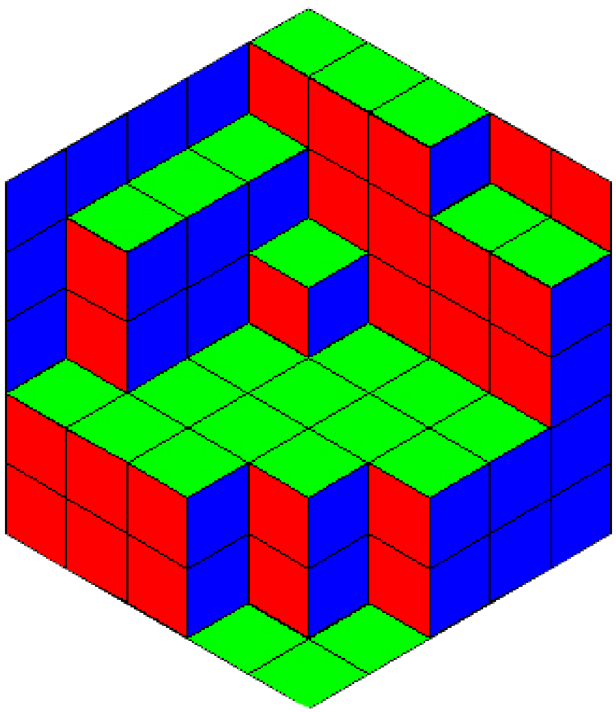
$$\text{(Dimers on } X) \longleftrightarrow \left(\begin{array}{l}\text{Tilings of } X^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells}\end{array}\right). \qquad \square$$

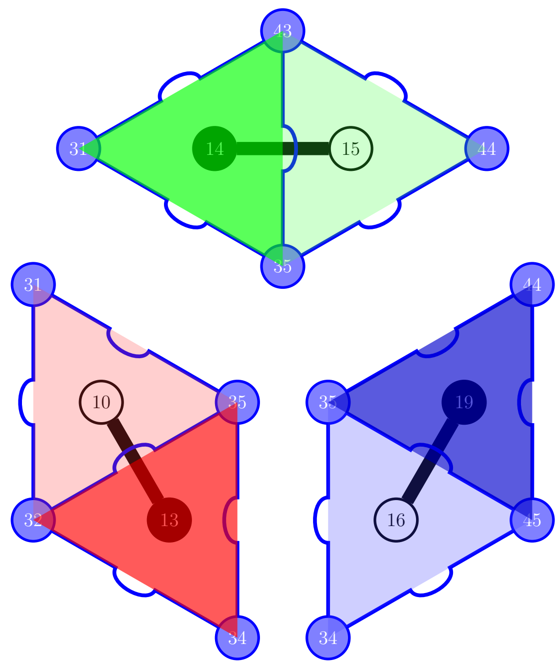
Remark. On bipartite graph, two-color tiles are admissible:

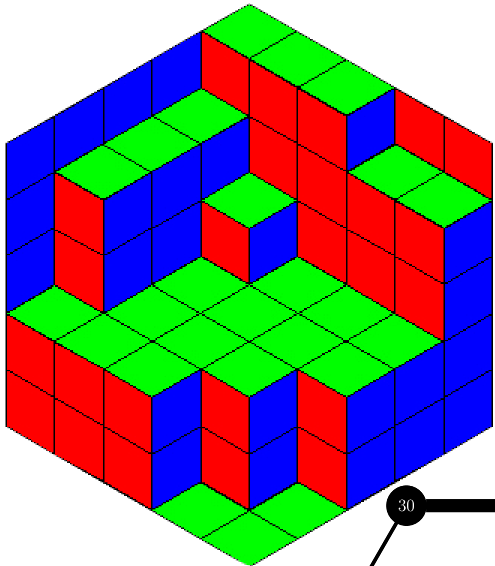




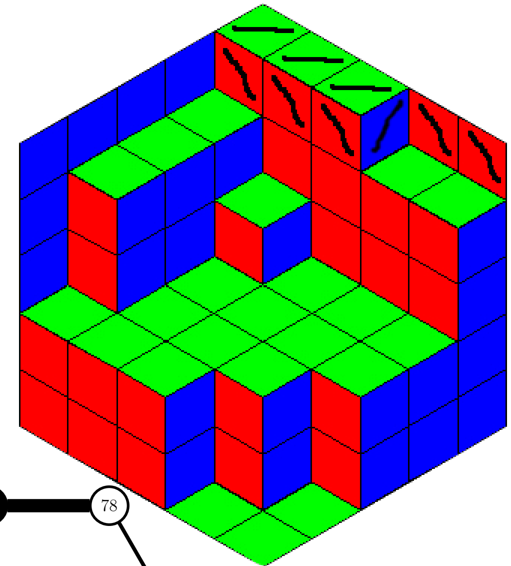
(Below: one-color tiles to the left, two-color tiles to the right)

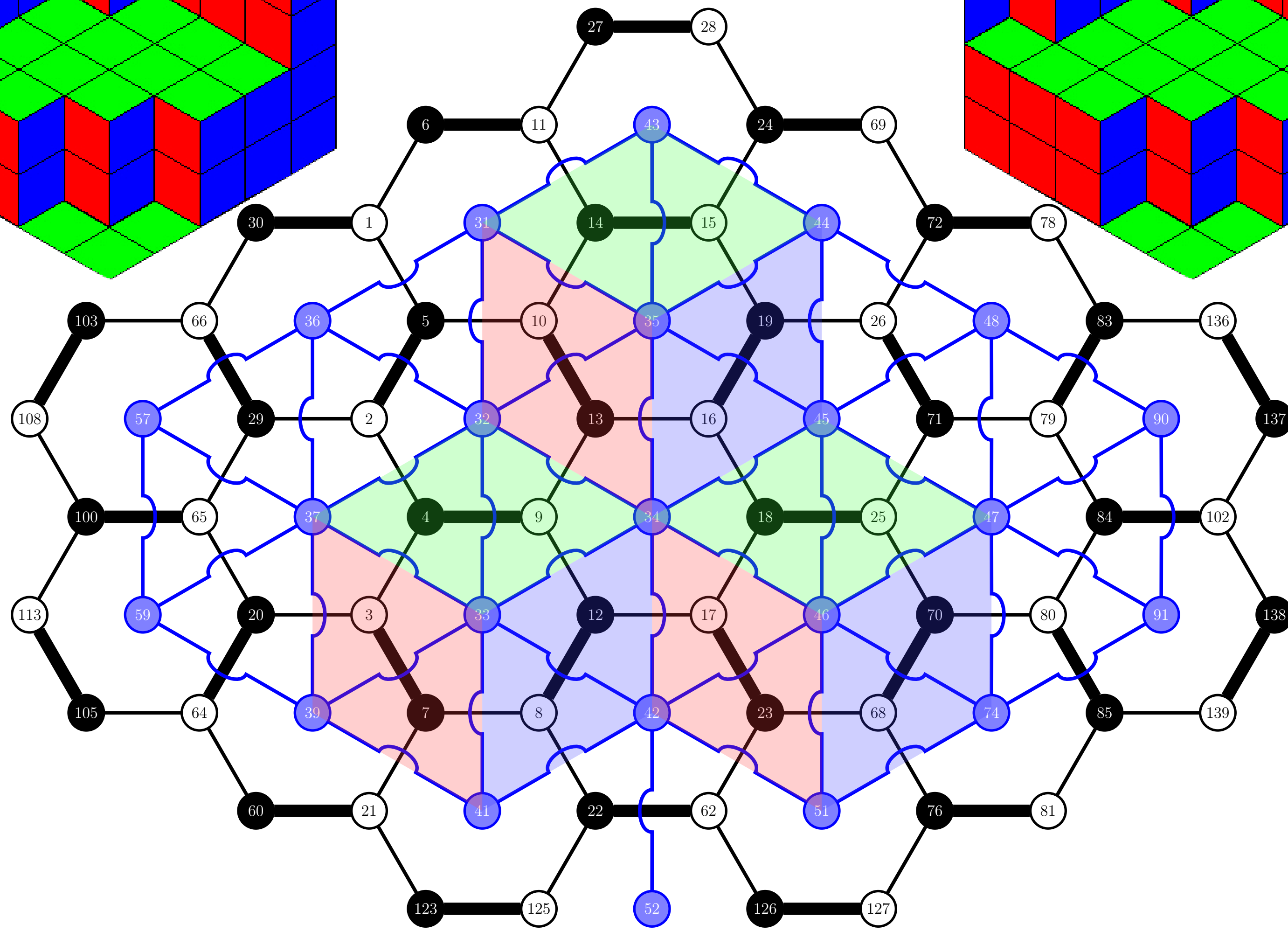






Cubes: 3D boxes out of 2D rhombus-tiling projection





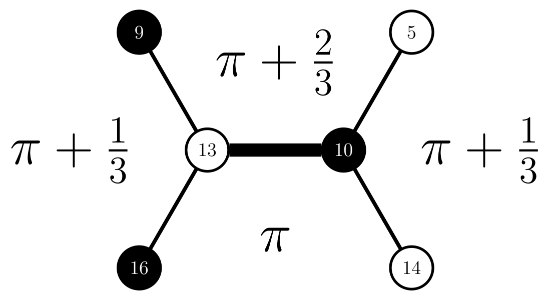
The height functions are parameterization space of spanning dual trees T^*:

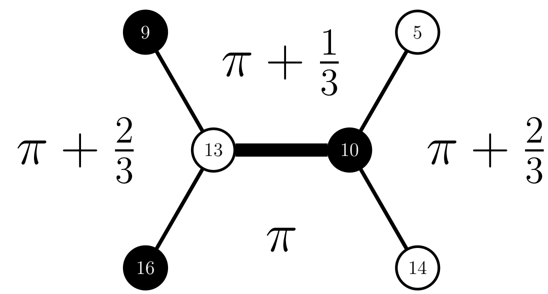
$$\mathcal{H}_X \overset{\text{def}}{=\!=} \{\pi : \mathscr{F}_X \longrightarrow \mathbb{Z}\}$$

on boundary-normalization $\pi(\mathscr{F}_0)=0 \mid f_0=$reference face, with respect to

$$\boldsymbol{Dimers} \longleftarrow\!\!| \ \boldsymbol{Discrete\ \ surfaces}.$$

In particular, in bipartite planar hexagonal $X \subset \mathbb{R}^2$:





Proposition (boundary-face).

(i) $\pi_D\big|_{\partial X} = \pi_D$ *restricted to boundary faces* ∂X *is independent of* D.

(ii) $\pi_{D_1 D_2} = \pi_{D_1} - \pi_{D_2}$.

Proof. $\heartsuit$.

1.1.1 What is known

Number of $\pm 1\cdot$ Pfaffians in Z for fixed $g \geqslant 0$

Kasteleyn (1963). For $g=0$, $Z = \pm$ Pfaffian of Kasteleyn matrix.

Kasteleyn (1963). For $g=1$, $Z =$ linear in 4 Pfaffians; 3 "$+$", 1 "$-$".

Kasteleyn (1963). For $g>1$, $Z =$ conjecture: Mysterious 2^{2g} Pfaffians; paper was unfinished, or at least unpublished.

Combinatorial representations of $\{+,-\}$ in Z

Gallucio & Loebl (1999). $Z := \pm 1$; $\overline{\mathcal{M}}_g$ compact orientable.

Tesla (2000). $Z := \sqrt{-1}$, ± 1; $\overline{\mathcal{M}}_g$ non-orientable; |Paffians| $\cong$ Kasteleyn.

Cimazoni & R. (2004, 2005). $Z := \pm 1$, by spin-structure model.

Cimasoni (2006). $Z := \sqrt{-1}$, by pin-minus structure for double-cover of $\overline{\mathcal{M}}_g$ non-orientable; $\cong$ spin structure's ± 1; a Tesla (2000) topological model.

Asymptotics of bipartite observables (Pfaffians)

R. et al. (2006). For bipartite, height functions $\pi(\mathscr{F})$, face-weights $q_{\mathscr{F}}$,

$$\boldsymbol{Z} \overset{\text{def}}{=\!=} \sum_{D\in\mathscr{D}} \prod_{\ell\in D} \omega_\ell \;=\; \text{Const.}\times \sum_{h}\prod_{\mathscr{F}} q_{\mathscr{F}}^{\pi(\mathscr{F})} \;\Big|\; q_{\mathscr{F}}^{\pi(\mathscr{F})} > 0,\; \pi : \mathscr{F}_X \longrightarrow \mathbb{Z}.$$

And, as $|X| \longrightarrow \infty$, $q_{\mathscr{F}} \longrightarrow 1$, in Seiberg-Witten conjecture (Gaussian field theory) entropy, $\boldsymbol{Z}$ equates to path integral of scaling limit:

$$\boldsymbol{Z} \;=\; \int \exp\left\{ -\frac{1}{2}\left(\int_{\overline{\mathcal{M}}_g} (\partial\Phi)^2\, d^2x \;+\; \int_{\overline{\mathcal{M}}_g} \lambda(x)\,\Phi(x) \right) \right\}$$

where all term $q_{\mathscr{F}}^{\pi(\mathscr{F})}$ contributes to the **R.H.S** linear multiple $\lambda(x)\,\Phi(x)$ by:

$$q_x \;=\; \ell^{-\varepsilon\cdot\lambda(x)} \quad \Big|\; \varepsilon = \text{lattice step};\; \lambda = \text{logarithmic scale, as } \varepsilon \longrightarrow 0.$$

Moreover, in Alvarez-Gaumé, Moore, Nelson & Vafa (1986), studying Fermi and Bose partition correspondence on Riemann surfaces,

$$\textbf{R.H.S.} \;\sim \sum_{\xi\in S(\overline{\mathcal{M}}_g)} \text{Arf}(\xi) \;\times\; |\Theta(z \mid \xi)|^2 \quad \Big|\; \omega \text{ determines } z.$$

Remark. Conjecture (critical weights): In large thermodynamic limit with scaling, the asymptotics of observables decaying linearly goes to

$$e^{\textsf{\textbf{Volume}}} \times \textsf{\textbf{the free energy}}$$

where next leading term is sum of theta functions; and, each theta function square is next leading asymptotics of each Pfaffian, respectively.

The conjecture was confirmed in:

(i) **Ferdinand (1967).** *On the square-grid torus.*

(ii) **Costa-Santos & McCoy (2002).** *For all $g \geqslant 2$, numerically by*

$$\mathsf{Arf}(\xi) \times |\Theta(z \mid \xi)|^2 .$$

That is, this conjecture works; but, still a conjecture i.e. no proof yet.

Remark. (i) Entropy model is preferred sophistication for fixed (not varying) genus, although observable equals derivative of logarithmic ω-system.

(ii) Z is glueable (summable) on boundary spins, for surfaces with boundary.

(iii) "Higher" spin-structure is unknown, perhaps a para-polynomial theory.

Goal

1. Operators

(i) Prove $\boldsymbol{Z}$, by genus g multi-edge bipartite manifold, for spanning T^*

(ii) Prove the $\mathcal{O}(n^3)$ observables for all fixed sufficient-large genus $g \geqslant 0$

2. Vertex algebras

(i) Prove graded kernel convergence for special genus g domain T^*

(ii) Find variational principle in thermodynamic $\ln(\cdot)$ scaling asymptotics

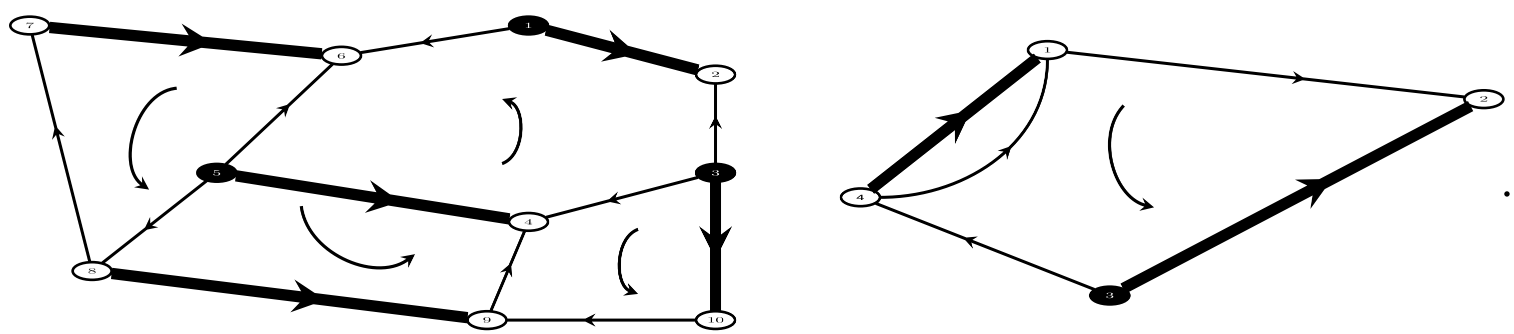
(iii) State a conjecture for the Green's function $\langle \cdot \rangle$ in large-deviation

1.1.2 Orientation in partition function

Definition. A compact orientable cell-decomposition (one-skeleton CW complex) $X \subset \overline{\mathcal{M}}_g$ is Kasteleyn X^K if by fixed (counterclockwise) boundary orientation $\varepsilon_{\partial\mathscr{F}} = \varepsilon_{\partial X}$, counter orientation $\varepsilon^-_{(\cdot)}$, edge orientation $\varepsilon^K_\ell \,|\, i_\ell \neq j_\ell$, odd parity $\rho^- = \mathbb{1}_{\mathscr{F}}(\varepsilon^K_\ell \in \varepsilon^-_{\varepsilon_{\partial\mathscr{F}}}) \pmod 2$

i.e. $\varepsilon^K_{\mathscr{F}} = \prod_{\ell \in \partial\mathscr{F}} \sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon^K_\ell) = -1, \ \forall \mathscr{F}$

$$\sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon^K_\ell) = \begin{cases} -1 & \text{if } \varepsilon^K_\ell \in \varepsilon^-_{\varepsilon_{\partial\mathscr{F}}} \\ +1 & \text{if } \varepsilon^K_\ell \in \varepsilon_{\partial\mathscr{F}}. \end{cases}$$



Then, given X^K such that ω_ℓ is trivial otherwise, $\forall\, \ell$ connecting i_ℓ and j_ℓ,

$$X^K_{ij} = \sum_\ell \varepsilon^K_{i_\ell j_\ell} \omega_\ell = -X^K_{ji} \quad \Big| \quad X^K_{ij} = 0\big|_{i=j}, \quad \varepsilon^K_{i_\ell j_\ell} = \begin{cases} -1 & \text{if } \varepsilon^K_\ell \text{ is } j_\ell \text{ to } i_\ell \\ +1 & \text{if } \varepsilon^K_\ell \text{ is } i_\ell \text{ to } j_\ell. \end{cases}$$

Remark. Bipartite Kasteleyn orientation is *well-defined* in hexagonal lattice.

Derivation. For simply connected bipartite $X \subset \overline{\mathcal{M}}_g$:

$$(X_{ij}^K) = \begin{cases} \text{Adjacency matrix if } \omega_\ell = 1,\ \forall\, \varepsilon_{i_\ell j_\ell}^K = \varepsilon_{j_\ell i_\ell}^K = 1 \\ \text{Weighted adjacency matrix if } \omega_\ell > 1,\ \forall\, \varepsilon_{i_\ell j_\ell}^K = \varepsilon_{j_\ell i_\ell}^K = 1 \end{cases}$$

with the twined

$$X_{ij}^K = -X_{ji}^K = \begin{cases} \omega_\ell & \text{if } i_\ell \bullet\!\!\longrightarrow\!\!\circ\, j_\ell \text{ or } i_\ell \bullet\!\!\longrightarrow\!\!\circ\, j_\ell \\ -\omega_\ell & \text{if } i_\ell \circ\!\!\longleftarrow\!\!\bullet\, j_\ell \text{ or } i_\ell \circ\!\!\longleftarrow\!\!\bullet\, j_\ell \\ 0 & \text{if } i_\ell, j_\xi \text{ such that } i_\ell = j_\ell \text{ or } \ell \neq \xi. \end{cases}$$

Lemma (equiv. classes). (i) $\{\sigma\,|_{\mathsf{Aut}(D)}\} \cong (\mathcal{S}_n \times \mathcal{S}_2^n)^{(\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n \times \mathcal{S}_2^n))}$

(ii) $|\{\widetilde{\sigma}\}| \leqslant \sqrt{(2n)! \cdot 2^{-((1/\varepsilon) \bmod c(X))} \cdot e^{\ln(a(X)\cdot b(X))}},\ \varepsilon > 0,\ n < \infty$

where

$$a, b, c \in \mathbb{R}^+,\ n \geqslant 2 \;\Big|\; \min(\deg(X)) \geqslant \frac{n! \cdot a(X) \cdot b(X)}{\lfloor 2n-3 \rfloor!! \;=\; \prod_{k=0}^{\lfloor n \rfloor - 2} 2k+1};$$

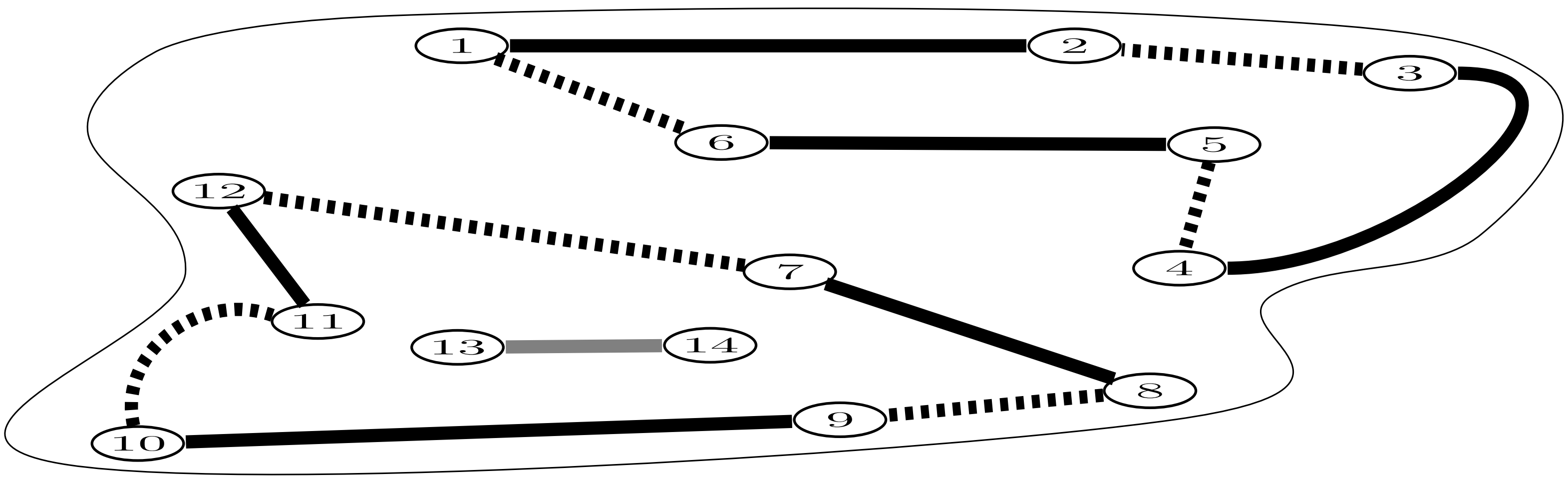
$$\mathsf{Im}(\mathsf{Aut}(\mathscr{D})) \longleftarrow X \ni \sigma := (\sigma(1), \ldots, \sigma(2n));\ \widetilde{\sigma} = \sigma \mid \sigma(2\ell) > \sigma(2\ell-1).$$

Proof. $\mathcal{S}_n := \{\, (\sigma(1), \ldots, \sigma(2n)),\ \ldots,\ (\sigma(2n-1), \sigma(2n), \ldots, \sigma(1), \sigma(2)) \,\}$
$\mathcal{S}_2^n := \{\, (\sigma(1), \ldots, \sigma(2n)),\ \ldots,\ (\sigma(2), \sigma(1), \ldots, \sigma(2n), \sigma(2n-1)) \,\}$
i.e. $|\{\widetilde{\sigma}\}| = |\sigma\,|_{\mathsf{Aut}(D)}| = |\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n \times \mathcal{S}_2^n)|$; $a(X) n! e^{\ln(b(X))} = \lfloor 2n-1 \rfloor!!$. $\square$

The transition subgraph is symmetry $D_1 \Delta D_2 = D_1 \cup D_2 \backslash D_1 \cap D_2$ of 1-chain complex $\mathcal{C}^1(X;\mathbb{Z}_2)$; 1-cycle in homology $\mathcal{H}^1(X;\mathbb{Z}_2) = \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$ class; all ordered even-length $\eta = \sum_{C_\alpha} \sigma_{D_1 \Delta D_2}(C_\alpha)$ simple closed transition paths $C_\alpha = (\sigma(n_{\alpha-1}+1), \ldots, \sigma(n_\alpha))$, $\forall\, \alpha \in \mathbb{N}^+ \mid 1 \leqslant \alpha \leqslant \eta$, $n_0 = 0$, traversing $\sigma(n_{\alpha-1}+1), \big(\sigma(n_{\alpha-1}+1), \sigma(n_{\alpha-1}+2)\big), \ldots, \sigma(n_\alpha), \big(\sigma(n_\alpha), \sigma(n_{\alpha-1}+1)\big)$:

$$\Big(\big(\sigma(n_{\alpha-1}+1), \sigma(n_{\alpha-1}+2)\big), \ldots, \big(\sigma(n_\alpha - 1), \sigma(n_\alpha)\big)\Big) \subseteq D_1$$

$$\Big(\big(\sigma(n_{\alpha-1}+2), \sigma(n_{\alpha-1}+3)\big), \ldots, \big(\sigma(n_\alpha), \sigma(n_{\alpha-1}+1)\big)\Big) \subseteq D_2\,.$$



Remark. D_1, D_2 are equivalent if $|D_1 \Delta D_2| = 0 \in \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$; $D_1, D_2 = $ 1-chain in cell-complex $\mathcal{C}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$; $\partial D_1, \partial D_2 = \mathcal{C}^0(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$.

Lemma (sign). *Monomial sign*

$$\varepsilon_D^K = (-1)^{t(\sigma)} \prod_{\ell \in D} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \;\Big|\; t(\sigma) := (\sigma(1), \ldots, \sigma(2n)) \longrightarrow (1, \ldots, 2n).$$

is invariant of Aut$(\mathscr{D})$.

Proof. ε_D^K is Aut(D) invariant by $(-1)^{t(\sigma)}$ and $\sigma(2\ell-1)\sigma(2\ell)$ transposition. Now, let $D_1, D_2 \in \mathscr{D}$ orient from $\sigma(2\ell-1)$ to $\sigma(2\ell)$, resp. $\tau(2\xi-1)$ to $\tau(2\xi)$, in cyclic order $\forall\, C_\alpha$, $\widetilde{\sigma}$, $\widetilde{\tau}$, in transition subgraph. Then, exactly one edge $\ell^* \vee \xi^*$ is $+$ $(-)$ on $\partial\mathscr{F}$ in clockwise (counterclockwise) C_α. But, $\forall\, \gamma = \sigma \circ \tau$ defined by $\sigma(2\nu-1)(2\nu) = \tau(2\nu-1)(2\nu)$,

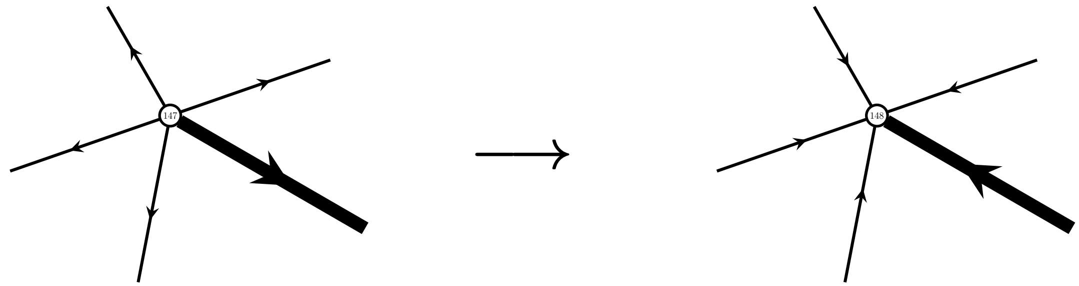
$$+1 = \varepsilon^K_{D_1}\,\varepsilon^K_{D_2} = \prod_{C_\alpha} \prod_{\ell \in C_\alpha} \prod_{\xi \in C_\alpha} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)}\, \varepsilon^K_{\tau(2\xi-1)\tau(2\xi)} \prod_{\nu} \varepsilon^K_{\gamma(2\nu-1)\gamma(2\nu)}$$

$$= \prod_{C_\alpha} \prod_{\ell \vee \ell^* \in C_\alpha} \prod_{\xi \vee \xi^* \in C_\alpha} \varepsilon^K_{\sigma(2(\ell \vee \ell^*)-1)\,\sigma(2(\ell \vee \ell^*))}\, \varepsilon^K_{\tau(2(\xi \vee \xi^*)-1)\,\tau(2(\xi \vee \xi^*))}$$

$\Longrightarrow \varepsilon^K_{D_1} = \varepsilon^K_{D_2}$, for $\mathbb{1}_{C_\alpha}(\varepsilon^K_{\ell \vee \ell^* \vee \xi \vee \xi^*} \in \varepsilon^-_{\varepsilon_{\partial X}}) = 1 \pmod 2$, $\forall\, \alpha$, by $\ell^* \vee \xi^*$

i.e. $\varepsilon^K_{D_1} = \varepsilon^K_{D_2}, \forall\, \rho^- = \mathbb{1}_{C_\alpha}\big(\varepsilon^K_{(\cdot)} \in \varepsilon^-_{\varepsilon_{\partial X}}\big) = \mathbb{1}_{C_\alpha}\big(\varepsilon^K_{(\cdot)} \in \varepsilon_{\partial X}\big) = \rho^+ \Longleftrightarrow$ Aut(D) invariance $\forall\, C_\alpha \Longleftrightarrow D_1, D_2 \in \mathscr{D} \Longleftrightarrow |\partial D| = 2n$, $\forall\, n \in \mathbb{N}^+$. $\square$

Definition. Two orientations are equivalent if there exist reversing-maps:



Lemma. *All Kasteleyn orientations of* $X \subset \mathbb{R}^2$ *are equivalent.*

Proof. By two Kasteleyn orientations K_- and K_+ marked as K_- (resp. K_+) on ith end (resp. jth end) of ℓ, $\forall\,\mathscr{F}$, with respect to $\varepsilon_{\partial\mathscr{F}} = \varepsilon_{\partial X}$,

$$+1 = \prod_{\ell\in\partial\mathscr{F}} \sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon_\ell^{K_-}) \prod_{\ell\in\partial\mathscr{F}} \sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon_\ell^{K_+}) = \prod_{\ell\in\partial\mathscr{F}} \sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon_\ell^{K_-}) \cdot \prod_{\ell\in\partial\mathscr{F}} \sigma_\ell^{K_-K_+} \cdot \sigma_{\varepsilon_{\partial\mathscr{F}}}(\varepsilon_\ell^{K_-})$$

where

$$\sigma_\ell^{K_-K_+} = \begin{cases} -1 & \text{if} \;\; \varepsilon_\ell^{K_-} \in \varepsilon^-_{\varepsilon_{\partial\mathscr{F}}},\, \varepsilon_\ell^{K_+} \in \varepsilon_{\partial\mathscr{F}} \;\; \text{or} \;\; \varepsilon_\ell^{K_-} \in \varepsilon_{\partial\mathscr{F}},\, \varepsilon_\ell^{K_+} \in \varepsilon^-_{\varepsilon_{\partial\mathscr{F}}} \\ +1 & \text{if} \;\; \varepsilon_\ell^{K_-},\, \varepsilon_\ell^{K_+} \in \varepsilon^-_{\varepsilon_{\partial\mathscr{F}}} \;\; \text{or} \;\; \varepsilon_\ell^{K_-},\, \varepsilon_\ell^{K_+} \in \varepsilon_{\partial\mathscr{F}} \end{cases}$$

i.e. $K_- \longleftrightarrow K_+ \longleftrightarrow$ equivalence class $[K]$ in simple orientation reversal around vertices, as required, by $\sigma_\ell^{K_-K_+} = -1$. □

Corollary. *Equivalence class* $[K]$ *is unique for* $X \subset \mathbb{R}^2$.

Proof. $\exists$ one homotopy class of loops i.e. $\mathbb{R}^2$ trivial fundamental group. □

Theorem. $|\{[K]\}| = 2^{2g} \mid [K] =$ Kasteleyn orientation equivalence class.

Proof. $\{[K]\} \cong$ non-degenerate affine closure of characteristic-2 field κ, skew symmetric quadratic form $\mathrm{Sym}^2_\kappa(V^\wedge)$:

$$q(\alpha+\beta) = q(\alpha)+q(\beta)+\alpha{\cdot}\beta \;\Big|\; q: V\otimes V \longrightarrow \kappa,\ \forall\,\alpha,\beta \in \mathcal{H}^1 = V\otimes V$$

for all $\alpha \in \mathcal{H}^1 =$ first homology space, classified by:

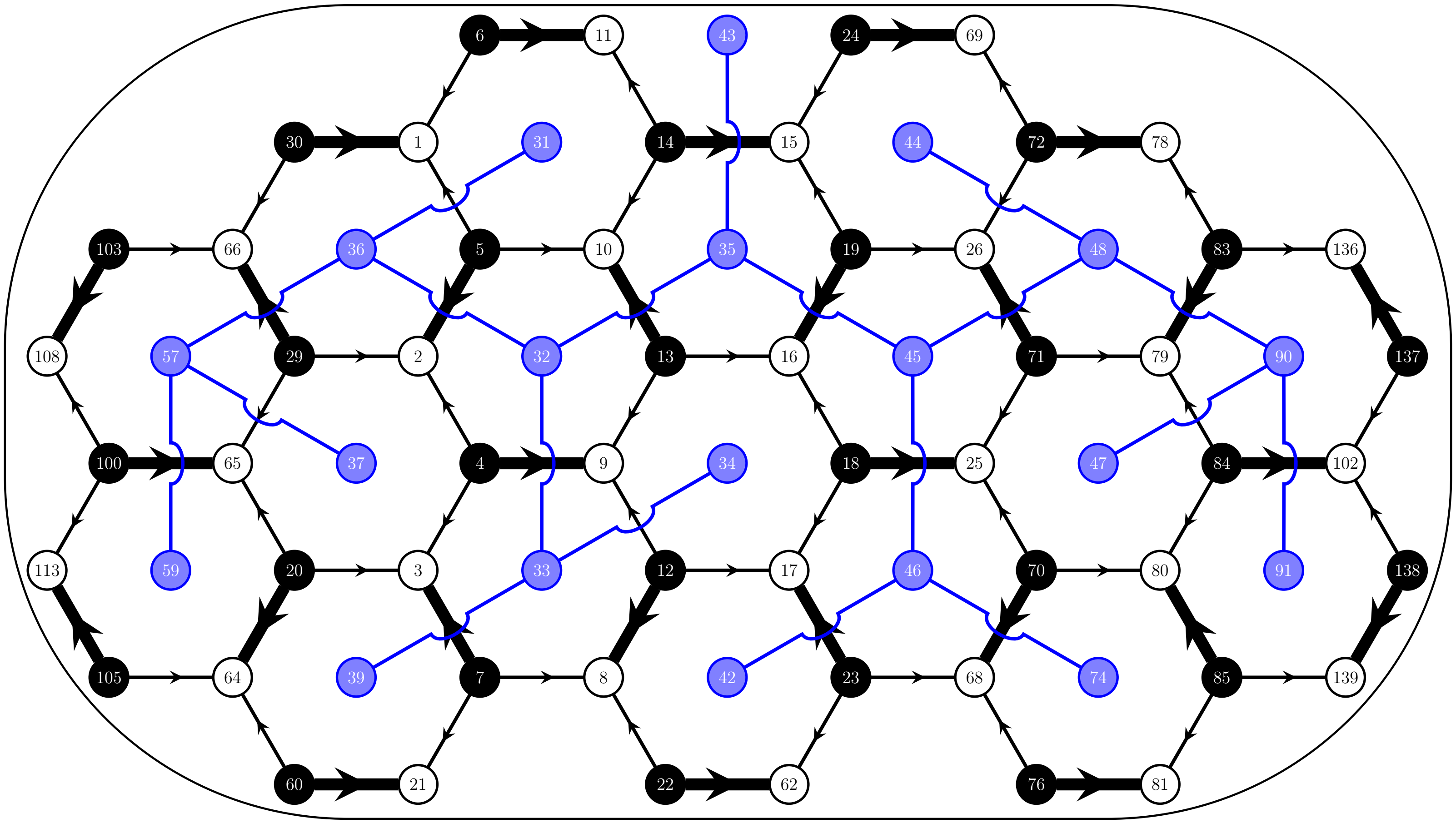
$$\frac{1}{\sqrt{|\mathcal{H}^1|}} \sum_{q\,\in\,(\mathcal{H}^1,\cdot)} (-1)^{\mathit{Arf}(q)+q(\alpha)} = 1 \;\mid\; \mathit{Arf}(q) = \sum_{\{\ell_i,\ell_j\}} q(\ell_i)q(\ell_j) \in \kappa/f(\kappa) \subset \mathbb{Z}_2$$

where $\{\ell_i, \ell_j\} =$ symplectic basis-pairs in symplectomorphisms $V \longrightarrow V$, for Lang's isogeny $f: \kappa \longrightarrow \kappa \mid x \longmapsto x^2 - x \in \mathrm{Gal}/\mathbb{F}_2$ (2-element Galois field).

Continuous map $\psi: X \longrightarrow \overline{\mathcal{M}}_g \mid X \supseteq \psi$-faces $\mathscr{F} \approx$ open disk $=$ connected components of $\overline{\mathcal{M}}_g \backslash \psi(X) \Longrightarrow \exists\ \chi(X) = \chi(\overline{\mathcal{M}}_g)$ in Euler-Poincaré bound $|V| - |E| + |\mathscr{F}| = \chi(X) \geqslant \chi(\overline{\mathcal{M}}_g)$. But, all vanishing composition $\partial_1 \circ \partial_2$ of boundary operators $\partial_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$, $\partial_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_0, \forall\, \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 =$ basis of 2D cell-complex vertices, edges, faces, resp., $\Longrightarrow$ 1-cycle space $\mathrm{Ker}(\partial_1)$ contains 1-boundary space $\partial_2(\mathcal{C}_2)$. Hence, independent of X, depending only on the genus g, $|\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)| = |\mathrm{Ker}(\partial_1)/\partial_2(\mathcal{C}_2)| = 2^{2g}$. □

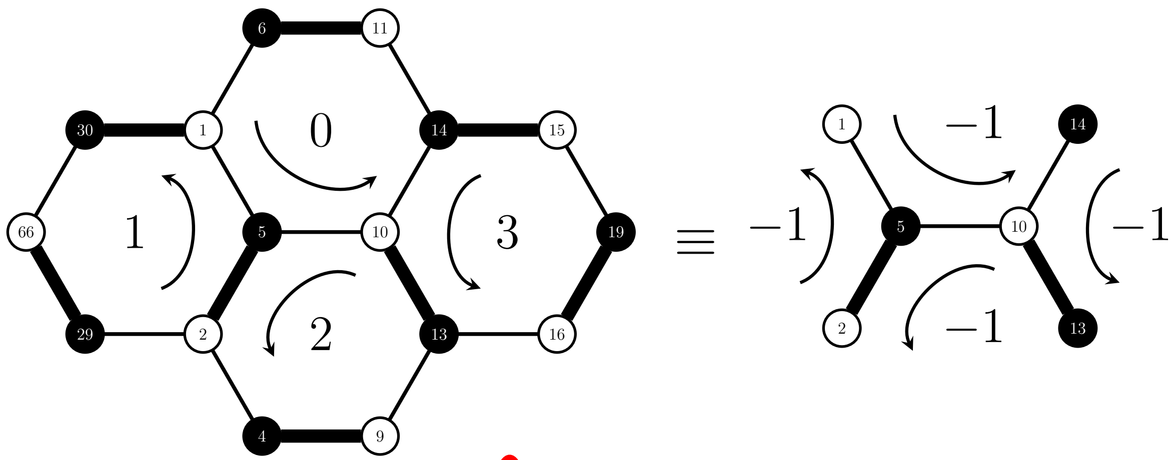
Lemma (existence). *Kasteleyn orientation exists.*

Proof. Following a rooted spanning dual tree T^*:

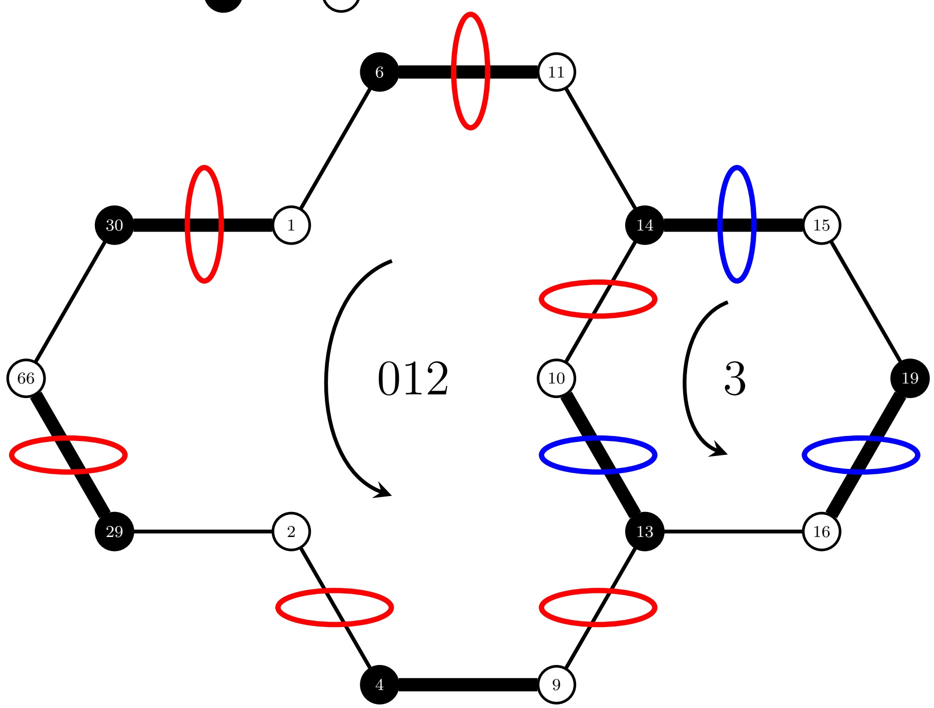


Reduce X to $\ll$: $n\times n \longrightarrow \exp(\alpha n^2)$. Arbitrarily orient every ℓ not crossing T^*. Deleting ℓ^* from leaves starting at root, make $\varepsilon^K_{\mathscr{F}}$, $\forall\,\mathscr{F}$. □

Remark. Deleted-vertex changes Kasteleyn to non-Kasteleyn at "hole":



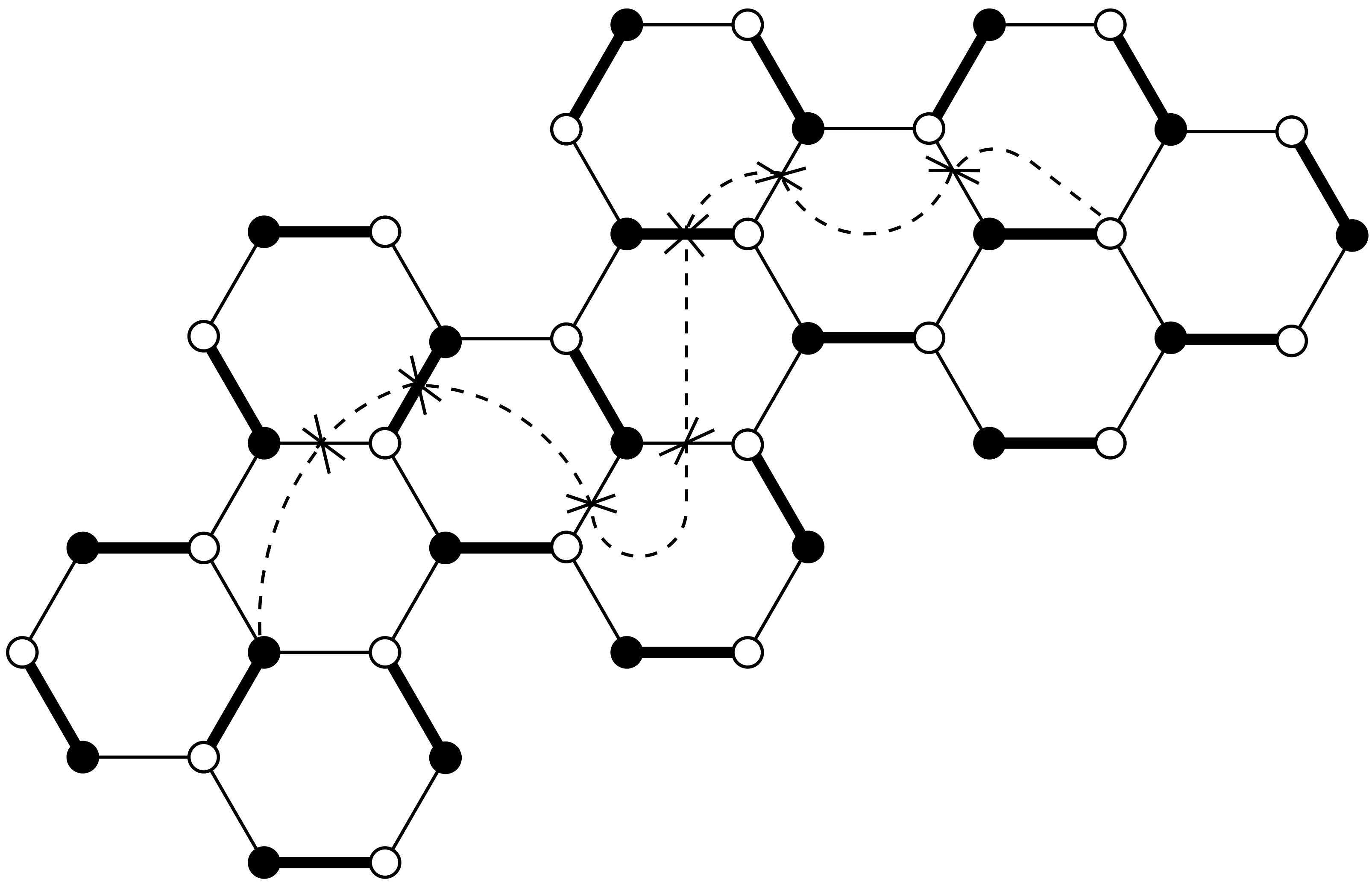
$h_0 = h_1 =$
$= h_2 = h_3 =$
$=$ Kasteleyn.



$h_{012} =$ non-Kasteleyn.

$h_3 =$ Kasteleyn.

Remark. To convert the non-Kasteleyn orientation back to Kasteleyn:



$$h_0 = h_1 = \cdots = h_{11} = -1.$$

Theorem. *For* $\mathbb{R}^n$*-valued* $X \subset \overline{\mathcal{M}}_g$ *of a fixed sufficiently large genus* g,

$$|\mathsf{Pf}(X^K)| = \boldsymbol{Z} \overset{\textit{def}}{=\!=} \sum_{D \in \mathscr{D}} \prod_{\ell \in D} \boldsymbol{\omega}_\ell$$

where

$$\mathbf{Quot}(\mathbb{K}[D]) \ni \mathsf{Pf}(X^K) = \frac{1}{n!} \frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \mathcal{S}_{2n}} \textit{sgn}(\boldsymbol{\sigma})\, X^K_{\boldsymbol{\sigma}(1)\boldsymbol{\sigma}(2)} \cdots X^K_{\boldsymbol{\sigma}(2n-1)\boldsymbol{\sigma}(2n)}$$

$$\textit{sgn}(\boldsymbol{\sigma}) = (-1)^{t(\boldsymbol{\sigma})} \;\mid\; t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}(1), \ldots, \boldsymbol{\sigma}(2n)) \longrightarrow (1, \ldots, 2n).$$

Proof. $X^K = m\times m \Longrightarrow \det X^K = \det(-(X^K)^T) = (-1)^m \det X^K = 0 \Longleftrightarrow m =$ odd; but $\det X^K \neq 0 \Longrightarrow \det X^K =$ positive-definite, square of rational function of $X^K_{ij} \mid X^K = 2n\times 2n$.

In particular, $X^K_{i\,\pi(i)} = -X^K_{\pi(i)\,i} \mid i \leqslant \pi(i) \Longrightarrow$ sum of 2-partition monomials:

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{\pi \\ \cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n)}} (-1)^{t(\pi)} \prod_{i=1}^{2n} X^K_{i\,\pi(i)} \quad \left| \begin{array}{l} j=\pi^{-1}(i) \longleftrightarrow i\neq j\in\{1,\ldots,n\} \\ \quad\Longrightarrow\ X^K_{i\,\pi(i)} \equiv X^K_{\pi(2\ell-1)\pi(2\ell)} \\ \quad\forall\,\ell=1,\ldots,n; \\ t(\pi) = \text{even (odd), for even } n \text{ (otherwise)} \\ t(\pi) := (\pi(1),\ldots,\pi(2n)) \longrightarrow (1,\ldots,2n) \end{array}\right. \\ \qquad + \\ \displaystyle\sum_{\substack{\pi \\ \cap \\ 2\cdot\left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}}\Big/\left(\mathcal{S}_2\times\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}-2\right\}}\right)\right)}} (-1)^{t(\pi)} \prod_{i=1}^{2n} X^K_{i\,\pi(i)} \quad \left| \begin{array}{l} j=\pi^{-1}(i) \longleftrightarrow i\neq j\in\{1,\ldots,n\} \\ \quad\Longrightarrow\ X^K_{i\,\pi(i)} \equiv X^K_{\pi(2\ell-1)\pi(2\ell)} \\ \quad\forall\,\ell=1,\ldots,n; \\ t(\pi) = \text{odd (even),} \\ \text{for even } n \text{ (otherwise).} \end{array}\right. \end{array}\right.$$

by Leibniz's second-index permutations.

And, $t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}(1), \ldots, \boldsymbol{\sigma}(2n)) \longrightarrow (1, \ldots, 2n)$ implies the quadratic:

$$\left\{ \begin{array}{l} \displaystyle\sum_{\substack{\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}} \\ \Cap \\ \mathcal{S}_{2n} / (\mathcal{S}_n \times \mathcal{S}_2^n)}} (-1)^{t(\boldsymbol{\pi}) + n + t(\boldsymbol{\sigma})} \left(\prod_{\ell \in D} X^K_{\boldsymbol{\sigma}(2\ell-1)\boldsymbol{\sigma}(2\ell)} \right)^2 \quad \left| \begin{array}{l} t(\boldsymbol{\pi}) = \text{even (odd)}, \\ \text{for even } n \text{ (otherwise)} \end{array} \right. \\ \qquad\qquad + \\ 2 \times \displaystyle\sum_{\substack{\left(\begin{array}{c} \boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}} \neq \boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}} \\ \Cap \\ \mathcal{S}_{2n} / (\mathcal{S}_n \times \mathcal{S}_2^n) \end{array} \right) \\ \cong \\ \left(\mathcal{S}_{\left\{ \frac{(2n)!}{n!\,2^n} \right\}} \Big/ \left(\mathcal{S}_2 \times \mathcal{S}_{\left\{ \frac{(2n)!}{n!\,2^n} - 2 \right\}} \right) \right)}} (-1)^{t(\boldsymbol{\sigma}) + t(\boldsymbol{\tau})} \prod_{\ell \in D} X^K_{\boldsymbol{\sigma}(2\ell-1)\boldsymbol{\sigma}(2\ell)} \prod_{\xi \in D} X^K_{\boldsymbol{\tau}(2\xi-1)\boldsymbol{\tau}(2\xi)} \end{array} \right.$$

$$= \left(\sum_{\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}}} (-1)^{t(\boldsymbol{\sigma})} \prod_{\ell \in D} X^K_{\boldsymbol{\sigma}(2\ell-1)\boldsymbol{\sigma}(2\ell)} \right)^2 = \mathsf{Pf}^2(X^K) \quad \left| \begin{array}{r} t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}(1), \ldots, \boldsymbol{\sigma}(2n)) \\ \longrightarrow (1, \ldots, 2n). \end{array} \right.$$

$\forall\, g, f \in \mathbb{R}^+, n \geqslant 2;\ \min(\deg(X)) \geqslant n!\, f(n) g(n) / \lfloor 2n-3 \rfloor!!;\ \mathsf{Aut}(\mathscr{D}) \subseteq \mathcal{S}_{2n}.$

Now, $\forall\,\xi$ connecting $\sigma(2\ell-1)$ and $\sigma(2\ell)$, and by ε_D^K invariant of $\mathrm{Aut}(\mathscr{D})$:

$$X^K_{\sigma(2\ell-1)\sigma(2\ell)} = \sum_{\xi\,\in\,(\sigma(2\ell-1),\sigma(2\ell))} \varepsilon^K_{\sigma(2\xi-1)\sigma(2\xi)}\ \omega_{\sigma(2\xi-1)\sigma(2\xi)}$$

$$\mathrm{Pf}(X^K) = \Big\{ \sum_{\sigma=\widetilde{\sigma}} \underbrace{\mathrm{sgn}(\sigma) \prod_{\ell\in D} \sum_{\xi\,\in\,(\sigma(2\ell-1),\sigma(2\ell))} \varepsilon^K_{\sigma(2\xi-1)\sigma(2\xi)}}_{\text{fixed, } \forall\,\sigma\,\in\,\mathrm{Aut}(\mathscr{D})}\ \omega_{\sigma(2\xi-1)\sigma(2\xi)}$$

$$= \Big\{ \sum_{\sigma\big|_{\mathrm{Aut}(D)}} \sum_D \underbrace{\mathrm{sgn}(\sigma) \prod_{\ell\in D} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)}}_{\text{fixed, } \forall\,\sigma\,\in\,\mathrm{Aut}(\mathscr{D})} \prod_{\ell\in D} \omega_\ell$$

$$= \Big\{ \frac{1}{n!}\frac{1}{2^n} \sum_{\substack{\sigma \\ \Cap \\ \mathrm{Aut}(D)}} \sum_D \underbrace{\varepsilon^K_D}_{\text{fixed, } \forall\,\sigma\,\in\,\mathrm{Aut}(\mathscr{D})} \prod_{\ell\in D} \omega_\ell$$

$$= \mathrm{sgn}(\sigma) \prod_{\ell\in D} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \cdot \sum_D \prod_{\ell\in D} \omega_\ell = (\pm) \sum_D \prod_{\ell\in D} \omega_\ell = \pm\,\boldsymbol{Z}$$

i.e.,

$$\mathsf{Pf}(X^K) = \sum_{\sigma|_{\mathsf{Aut}(D)}} \mathsf{sgn}(\sigma) \prod_{\ell \in D} X^K_{\sigma(2\ell-1)\sigma(2\ell)} \quad \Big| \; |\mathsf{Pf}(X^K)| = \boldsymbol{Z}$$

therefore, such that all $\mathcal{S}_{2n} \backslash \mathsf{Aut}(\mathscr{D})$ monomials vanish, the polynomial

$$\mathsf{Pf}(X^K) = \frac{1}{n!} \frac{1}{2^n} \sum_{\sigma \in \mathcal{S}_{2n}} \mathsf{sgn}(\sigma)\, X^K_{\sigma(1)\sigma(2)} \cdots X^K_{\sigma(2n-1)\sigma(2n)} \quad \Big| \; |\mathsf{Pf}(X^K)| = \boldsymbol{Z}$$

which differs only by orientation, independent of $\sigma \in \mathsf{Aut}(\mathscr{D})$. □

Theorem. *The observable is absolutely continuous iff X^K is non-singular.*

Proof.

$$\left\langle \prod_{i=1}^{k} \sigma_D(i_\ell j_\ell) \right\rangle = \mathsf{Pf}\big((X^K)^{-1}_{\xi\eta}\big) \quad \left| \begin{array}{l} D \ni (i_1 j_1), \ldots, (i_k j_k); \;\; \xi, \eta = 1, \ldots, k \\ |\mathsf{Pf}(X^K)| = \text{partition function.} \end{array} \right.$$ □

Remark. Combinatorial (exponential) is reduced to cubic complexity, since $\mathsf{Pf}(\mathcal{A} X^K \mathcal{A}^T) = \det(\mathcal{A}) \mathsf{Pf}(X^K) \longrightarrow \mathcal{O}(n^3)$, diagonalizing by skew symmetric Gaussian elimination; the pointwise-determined g behavior is universal.

Theorem. Prob$(D) = \frac{1}{\boldsymbol{Z}} \prod_{\mathscr{F}} q_{\mathscr{F}}^{\pi_D(\mathscr{F})}$, $\boldsymbol{Z} = \sum_D \prod_{\mathscr{F}}(\cdot)$; *giving measure*

$$\text{Prob}(\pi) = \frac{1}{\boldsymbol{Z}} \prod_{\mathscr{F}} q_{\mathscr{F}}^{\pi(\mathscr{F})}, \ \boldsymbol{Z} = \sum_{\pi \in \mathcal{H}_X} \prod_{\mathscr{F}} (\cdot), \ q_{\mathscr{F}} = \prod_{\ell \in \partial\mathscr{F}} \omega_\ell^{\sigma_{\varepsilon_{\partial X}}(\varepsilon_\ell^K)} \Big| \pi : \boldsymbol{\mathscr{F}}_X \longrightarrow \mathbb{Z}.$$

Proof. By the combinatorial bijection (equivalence) $\forall\, T^*$,

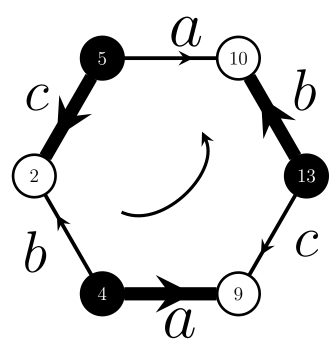
$$\{\textit{Dimers on } X\} \underset{\textit{bijection}}{\cong} \{\textit{height functions}\}.$$

Hence, Prob(D) = Prob(π) follows by the boundary-face proposition. □

Remark. Prob(D) = "gauge" invariant measure: $\omega_\ell \longmapsto s(\ell_+)\ \omega_\ell\ s(\ell_-)$. Furthermore, $q_{\mathscr{F}}$ = invariant ("essential" parameters).

Cases.

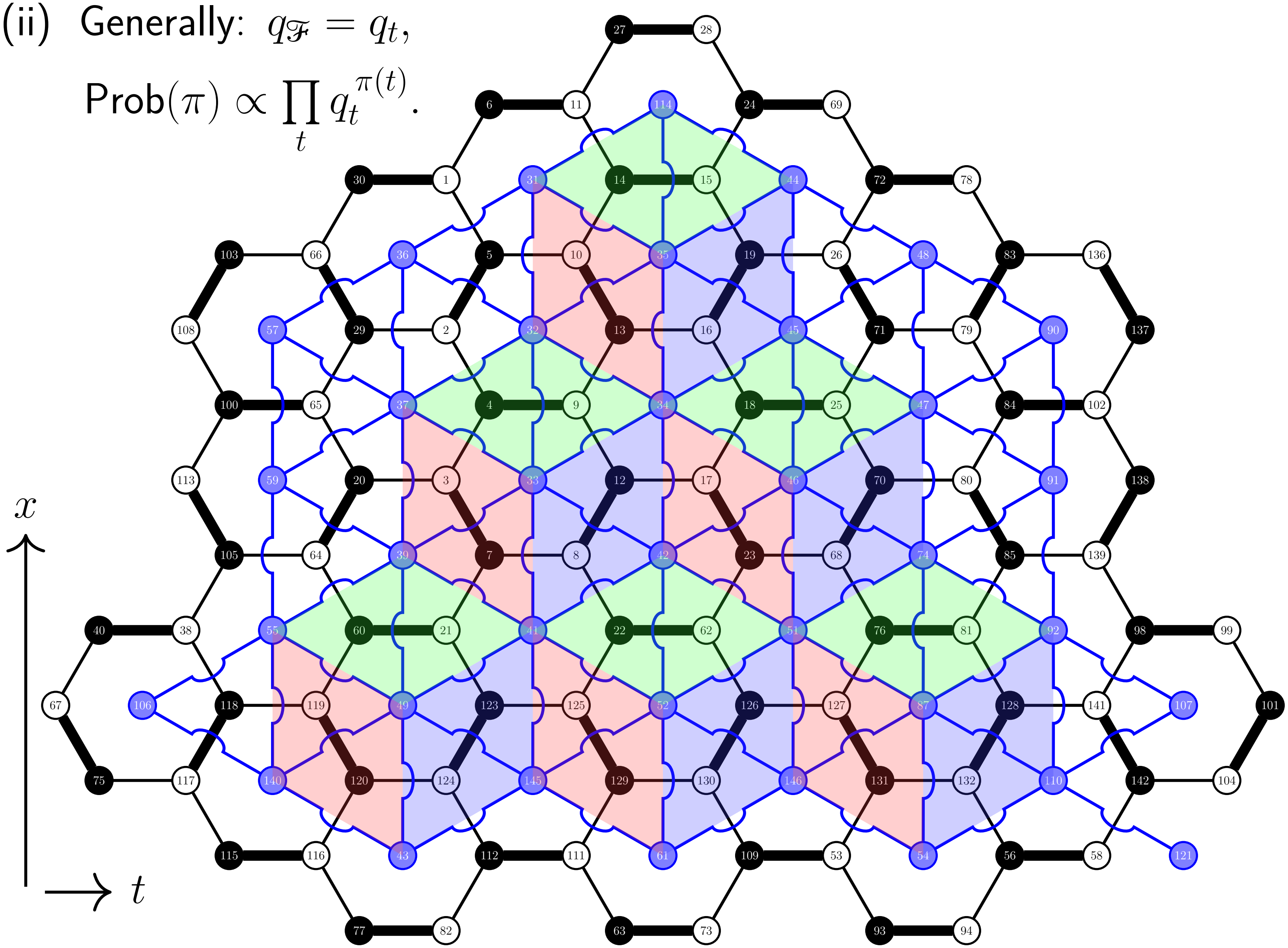
(i) Uniform distribution:



$$q = a^{-1} b\, c^{-1} a\, b^{-1} c = 1.$$

(ii) Generally: $q_{\mathscr{F}} = q_t$,

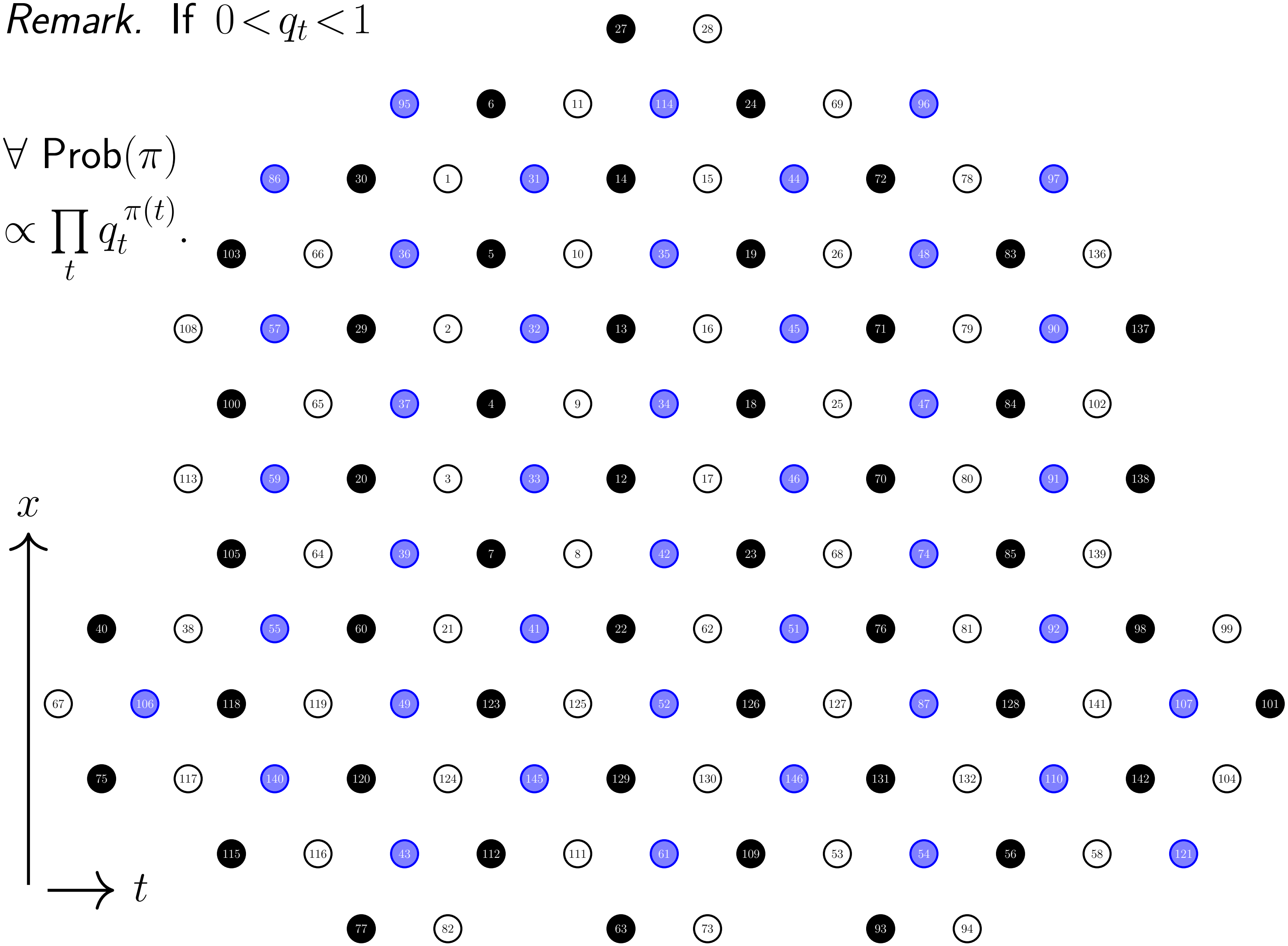
$$\mathrm{Prob}(\pi) \propto \prod_t q_t^{\pi(t)}.$$



Remark. If $0 < q_t < 1$

$\forall$ Prob(π)

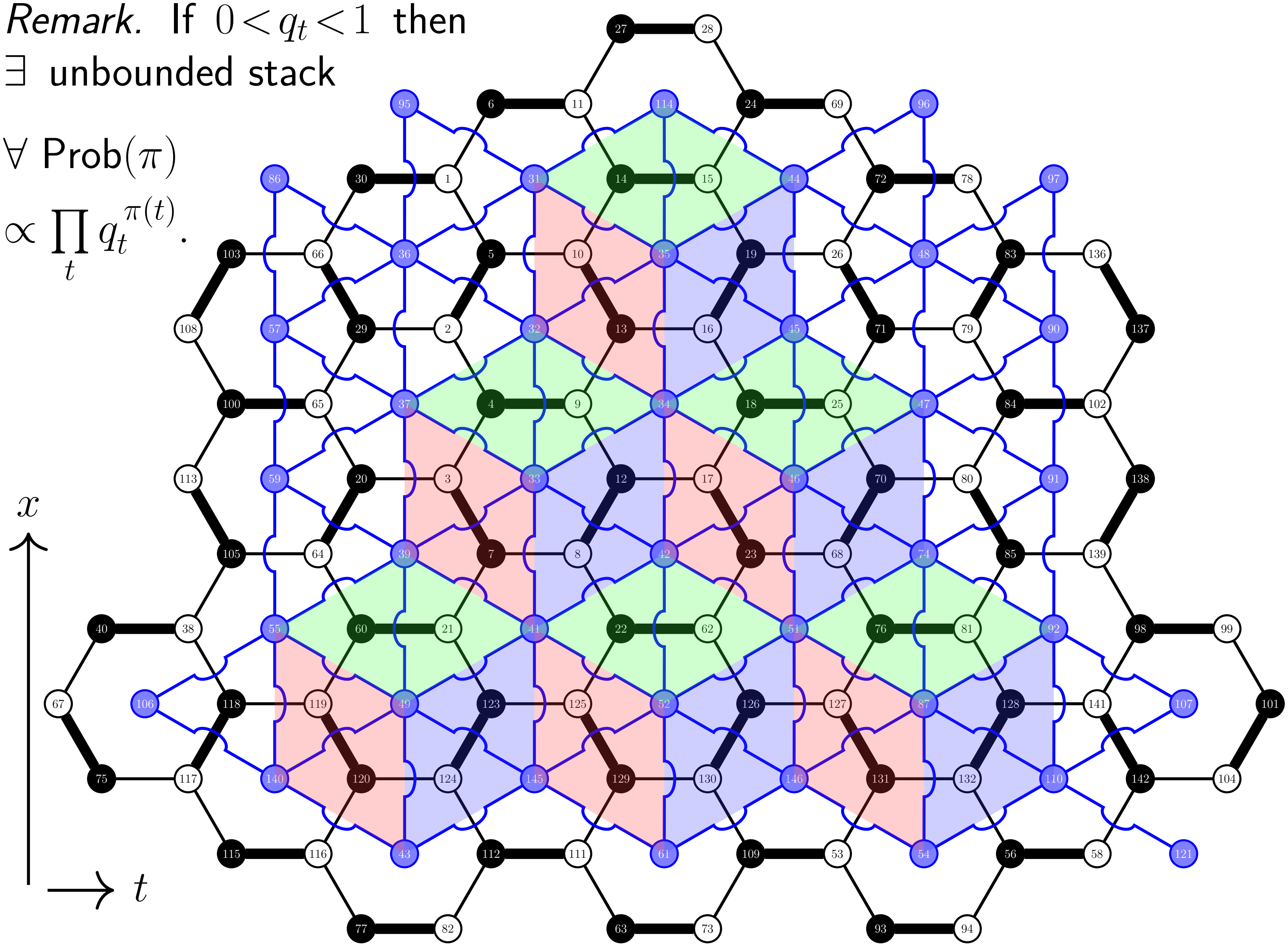
$\propto \prod_t q_t^{\pi(t)}$.



Remark. If $0 < q_t < 1$ then $\exists$ unbounded stack

$\forall$ Prob(π)

$\propto \prod_t q_t^{\pi(t)}$.



1.1.3 Graded (Grassmann) integral

Definition. Graded (Grassmann) algebra $\bigwedge^{\bullet} X$ on X basis $(e_1, \ldots, e_{2n})$ is generated by $2^{2n} = \sum_{k=0}^{2n} (\dim \bigwedge^k X) = \sum_{k=0}^{2n} \binom{2n}{k}$ dimensional basis vectors

$$\left\{ \begin{array}{l} e_0 = 1;\ e_{\sigma(k)<} = e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)} \ \big|\ e_{\sigma(\xi)} \otimes e_{\sigma(\eta)} + e_{\sigma(\eta)} \otimes e_{\sigma(\xi)} = 0; \\ \sigma(k)< = (\sigma(1), \ldots, \sigma(k)),\ \forall\, 1 \leqslant \sigma(1) < \cdots < \sigma(k) \leqslant k = 1, \ldots, 2n \end{array} \right\}.$$

Element is graded by a scalar and $\binom{2n}{k}$ k-vectors in $\bigwedge^k X \,|\, k = 1, \ldots, 2n$:

$$X_0 \oplus \bigoplus_{k=1}^{2n} \sum_{\sigma \in \mathcal{S}_{\sigma(k)<}} (-1)^{t(\sigma)} X_{\sigma(1) \cdots \sigma(k)}\, e_{\sigma(k)<} \ \Bigg|\ \begin{array}{r} t(\sigma) := (\sigma(1), \ldots, \sigma(k)) \\ \longrightarrow \sigma(k)< . \end{array}$$

Multiplication is by $\bigwedge^k X, \bigwedge^l X :\ X_0\big|_k \cdot X_0\big|_l = X_0\big|_{kl}$ and, $\forall\, k, l = 1, \ldots, 2n$,

$$X_{\sigma(1) \cdots \sigma(k)} \cdot X_{\tau(1) \cdots \tau(l)} =$$
$$= X_{\sigma(1) \cdots \sigma(k)} \cdot X_{\tau(1) \cdots \tau(l)} \bigotimes_{i=1}^{k} e_{\tau(i)} \otimes \bigotimes_{j=1}^{l} e_{\tau(j)} \ = \ 0 \Longleftrightarrow \sigma(i)\big|_k = \ \sigma(j)\big|_l .$$

Derivation. $\bigwedge^k : \bigotimes^k \longrightarrow \bigotimes^k \,|\, X_{ij} = -X_{ij};\ X_{\sigma(1) \cdots \sigma(k)} = \prod_{i=1}^{k/2} X_{\sigma(2i-1)\sigma(2i)}$:

$$e_{\sigma(k)<} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_{\sigma(k)<}} (-1)^{t(\sigma)} \bigotimes_{i=1}^{k} e_{\sigma(i)}\, ;\ X_{\sigma(k)<} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_{\sigma(k)<}} (-1)^{t(\sigma)} X_{\sigma(1) \cdots \sigma(k)} .$$

Derivation.

$$\bigwedge^2 X \ni x^{(1)} = \sum_{\sigma(i),\sigma(j)} X_{\sigma(i)\,\sigma(j)}\; e_{\sigma(i)} \otimes e_{\sigma(j)} \quad \Big|\; X_{\sigma(1)\cdots\sigma(k)} = \prod_{i=1}^{k/2} X_{\sigma(2i-1)\sigma(2i)}$$

$$\bigwedge^{2n} X \ni x^{(n)} = \mathsf{Pf}(A)\; e_{\sigma(2n)<} \quad \Big|\; \sigma(i), \sigma(j) = 1, \ldots, 2n\,.$$

Definition. With respect to orientation $x \in \bigwedge^{2n} X \simeq \mathbb{R}$, integral on $\bigwedge^{\bullet} X$ is

$$\int_{\bigwedge^{2n} X} f = f_x \quad \Big|\; f = f_x\, x + \underbrace{\cdots}_{\text{lower order terms}}\,.$$

In particular, if (x_i) is basis in V, then $x = x_1 \otimes \cdots \otimes x_n$ such that:

(i) $$\int \bigotimes_{i=1}^{k} x_{\sigma(i)} \otimes dx = \begin{cases} (-1)^{t(\sigma)} & \text{if } k = 2n \\ 0 & \text{if } k < 2n \end{cases} \quad \Bigg|\; \begin{aligned} & dx \cong (-1)^{n(2n-1)} \bigotimes_{i=1}^{2n} dx_i \\ & t(\sigma) := (\sigma(1), \ldots, \sigma(k)) \\ & \qquad \longrightarrow \sigma(k)< \,. \end{aligned}$$

(ii) $$\int \bigotimes_{i=1}^{2n} x_i \otimes \bigotimes_{j=1}^{2n} dx_j = (-1)^{n(2n-1)} \int \bigotimes_{i=1}^{2n} \Big(x_i \otimes dx_i \Big) = (-1)^{n(2n-1)}.$$

Lemma. *$\bigwedge^\bullet V$ graded identity, up to tensors on superalgebra $M_{a,b}$ minimal subfield, is isomorphic to kernel of either $\mathbb{Q}$ or prime-ordered field $\mathbb{F}_{q=p^m}$.*

Proof. $\heartsuit$.

Theorem. *T-ideal of $M_{pr+qs,\, ps+qr}$ is contained in T-ideal of $M_{p,q} \otimes M_{r,s}$.*

Proof. Follows from the prior lemma.

Theorem. *Let $A^*(a) = \int_{\bigwedge^\bullet V} A(a) \; \big| A(a) = \exp\big(\frac{1}{2} \sum_{ij} a_i A_{ij} a_j\big)$, $(a_i) \subseteq V$. Then A^* uniquely maximizes $-\int_{\bigwedge^\bullet V} A \log A$ such that:*

(i) $\mathsf{Pf}(A) = \displaystyle\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\Big) da$

(ii) $\mathsf{Pf}\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = \det(A)$

(iii) $(\mathsf{Pf}(A))^2 = \det(A)$

(iv) $\dfrac{\partial}{\partial A_{i_1 j_1}} \cdots \dfrac{\partial}{\partial A_{i_k j_k}} \mathsf{Pf}(A) = \mathsf{Pf}(A) \cdot \mathsf{Pf}((A^{-1})_{ab}) \;\Big|_{\substack{a = i_1, \ldots, i_k \\ b = j_1, \ldots, j_k}}.$

Proof (hints).

(i). Write:

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)\, da \;=\; \frac{1}{n!}\frac{1}{2^n}\int_{\bigwedge^\bullet V} \langle a, Aa\rangle^n\, da$$

such that

$$\int \langle a, Aa\rangle^{2n}\, da \;=\; \int a_{\sigma(1)}a_{\tau(1)}\cdots a_{\sigma(n)}a_{\tau(n)}\, A_{\sigma(1)\,\tau(1)}\cdots A_{\sigma(n)\,\tau(n)}\, da \;=$$

$$= \;(-1)^{t(\sigma)}A_{\sigma(1)\,\tau(1)}\cdots A_{\sigma(n)\,\tau(n)} \qquad \Big|\; t(\sigma):\; (\sigma(1),\tau(1),\ldots,\sigma(n),\tau(n)) \longrightarrow (1,\ldots,2n).$$

This implies

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)\, da \;=\; \frac{1}{n!}\frac{1}{2^n}\,\mathsf{Pf}(A).$$

Remark. II, III and IV follow from the latter integral formula.

(ii). Choosing splitting $V = W \oplus W^*$ by matrix block structure, where V is isomorphic to algebra (tensor product) generated by $c_i, b_i \,|\, i = 1, \ldots, n$ with relations $c_i c_j = -c_j c_i$, $c_i b_j = -b_j c_i$, and $b_i b_j = -b_j b_i$:

$$(a_1, \ldots, a_{2n}) = \\ = (\underbrace{c_1, \ldots, c_n}_{\text{basis in } W}, \underbrace{b_1, \ldots, b_n}_{\text{basis in } W^*}).$$

As a result,

$$\left\langle a\,, \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} a \right\rangle = 2\,\langle c, Ab\rangle.$$

Therefore, prove

$$\int\limits_{\bigwedge^n (W \oplus W^*)} \exp\left(\langle c, Ab\rangle\right) dc\, db = \det(A).$$

(iii). Similar.

(iv). $$\int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle a, \eta\rangle\Big)da \quad =$$

$$= \quad \int \exp\Big(\frac{1}{2}\big\langle a{+}A^{-1}\eta\,,\; A(a{+}A^{-1}\eta)\big\rangle \;-\; \frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big)da$$

$$= \quad \mathsf{Pf}(A)\;\exp\Big(-\frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big).$$

$$\frac{\partial}{\partial A_{i_1 j_1}}\cdots\frac{\partial}{\partial A_{i_k j_k}}\,\mathsf{Pf}(A) \quad =$$

$$= \quad \int a_{i_1}a_{j_1}\cdots a_{i_k}a_{j_k}\exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)da$$

$$= \quad \Big(\frac{\partial}{\partial \eta}\Big)^{2k}\int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle \eta, a\rangle\Big)da. \qquad \square$$

Theorem. *Given bipartite* $X^K \subset \mathbb{R}^2$,

$$Z_{X^K} = \varepsilon_X^K \int \exp\Big(\frac{1}{2}\sum_{ij} a_i\,(X_{ij}^K)\,a_j\Big)da \quad \Bigg|\begin{array}{l}\varepsilon_X^K = (-1)^{\sigma}\,\varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{2n-1}\sigma_{2n}}\\ 2n = |V(X^K)|.\end{array}$$

Proof. X^K bipartite $V(X^K) = V_\bullet(X^K) \sqcup V_\circ(X^K)$ implies

$$X^K = \begin{pmatrix} 0 & B_{X^K} \\ -(B_{X^K})^T & 0 \end{pmatrix} \quad \left|\begin{array}{l} B_{X^K}:\ \mathbb{R}^{V_\circ(X^K)} \longrightarrow \mathbb{R}^{V_\bullet(X^K)} \\ \mathbb{R}^{V(X^K)} = \ \mathbb{R}^{V_\bullet(X^K)} \oplus \mathbb{R}^{V_\circ(X^K)} \\ \dim(\mathbb{R}^{V_\bullet(X^K)}) = \dim(\mathbb{R}^{V_\circ(X^K)}) = n \\ |V(X^K)| = 2n. \end{array}\right.$$

Identifying $V_\bullet(X^K)$, $V_\circ(X^K)$ via a diagram $\{b\} \sim \{\omega\}$ with "hole"

$$X^K = \begin{pmatrix} 0 & C_{X^K} \\ -(C_{X^K})^T & 0 \end{pmatrix} \quad \left|\begin{array}{l} \mathbb{R}^{V(X^K)} = \ \mathbb{R}^{V_\bullet(X^K)} \oplus \mathbb{R}^{V_\circ(X^K)} \hookleftarrow \\ C_{X^K} = \ \mathbb{R}^{V_\circ(X^K)} \hookleftarrow \\ \hookleftarrow \ \Longrightarrow \ \text{recursion i.e. nested.} \end{array}\right.$$

That is, $Z_{X^K} = |\det(C_{X^K})|$. □

Corollary. *For all bipartite Kasteleyn observables,*

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle = \frac{\partial}{\partial\, \omega(b_1 w_1)} \cdots \frac{\partial}{\partial\, \omega(b_k w_k)} \ln Z_{X^K}$$

$$= \det\Big(\big((C_{X^K})^{-1} \big)_{\widetilde{b}\, w} \Big) \;\Bigg|\begin{matrix} \widetilde{b} = \widetilde{b}_1, \dots, \widetilde{b}_k \\ w = w_1, \dots, w_k \end{matrix}$$

where $\widetilde{b}$ = white-vertex identified with b.

Proof. $\heartsuit$.

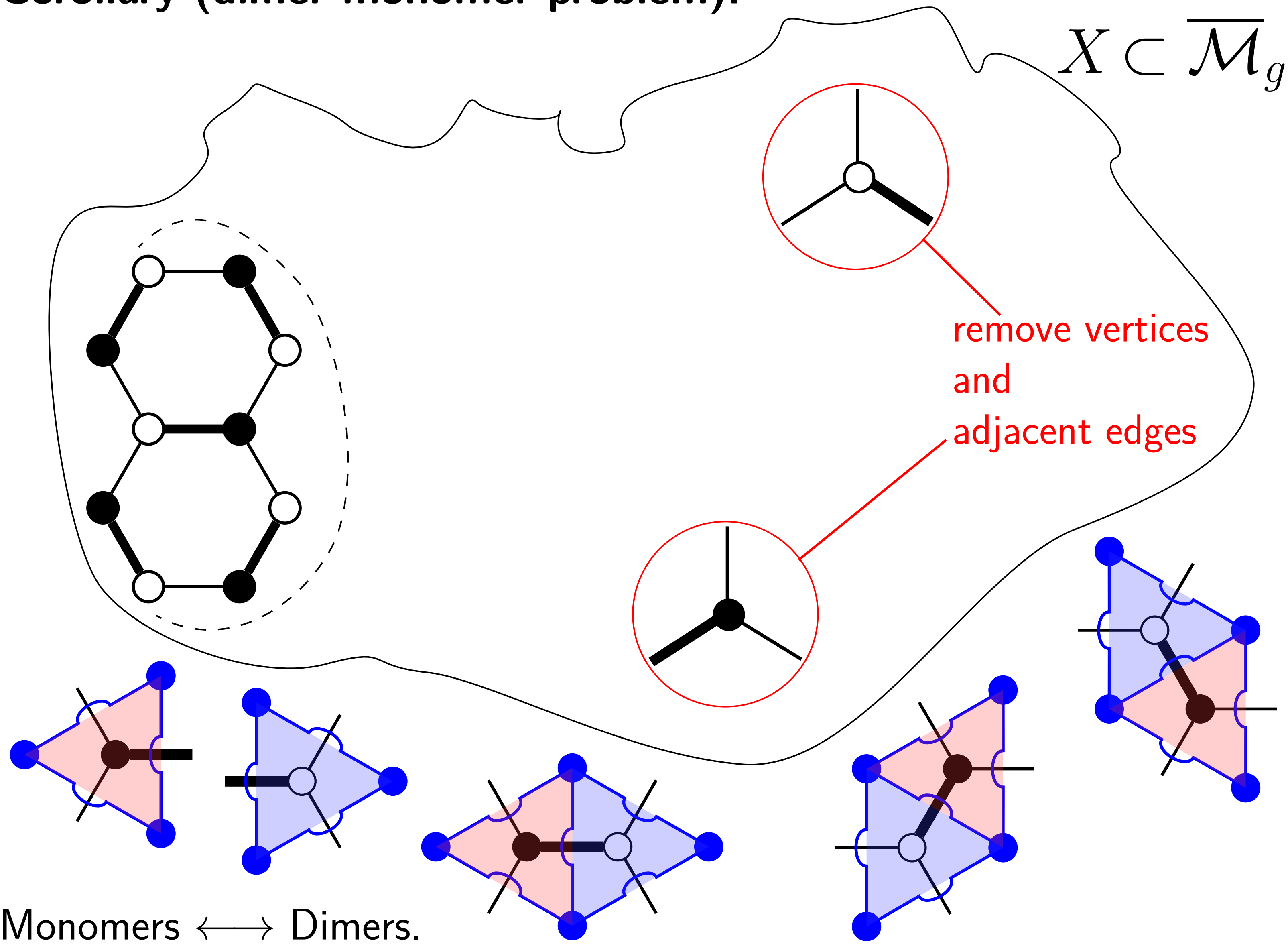
Remark. The "physical" meaning:

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle =$$

$$= \int \psi^*_{b_1} \psi_{w_1} \cdots \psi^*_{b_k} \psi_{w_k} \exp\big(\psi^* C_{X^K}\, \psi \big) d\psi^* d\psi \cdot \int \exp\big(\psi^* C_{X^K}\, \psi \big) d\psi^* d\psi$$

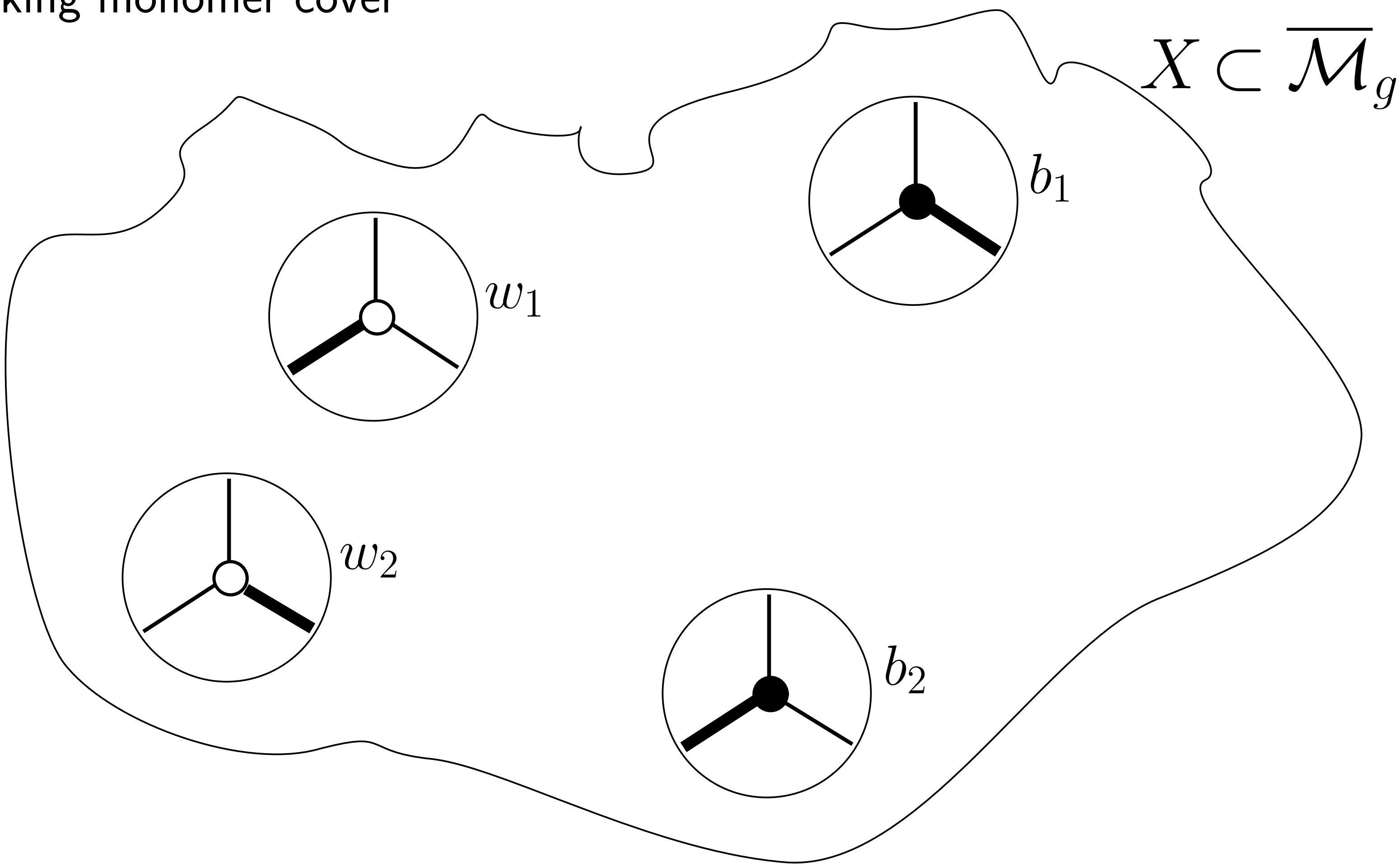
which corresponds to the free Fermionic observables.

Corollary (dimer-monomer problem).



Monomers $\longleftrightarrow$ Dimers.

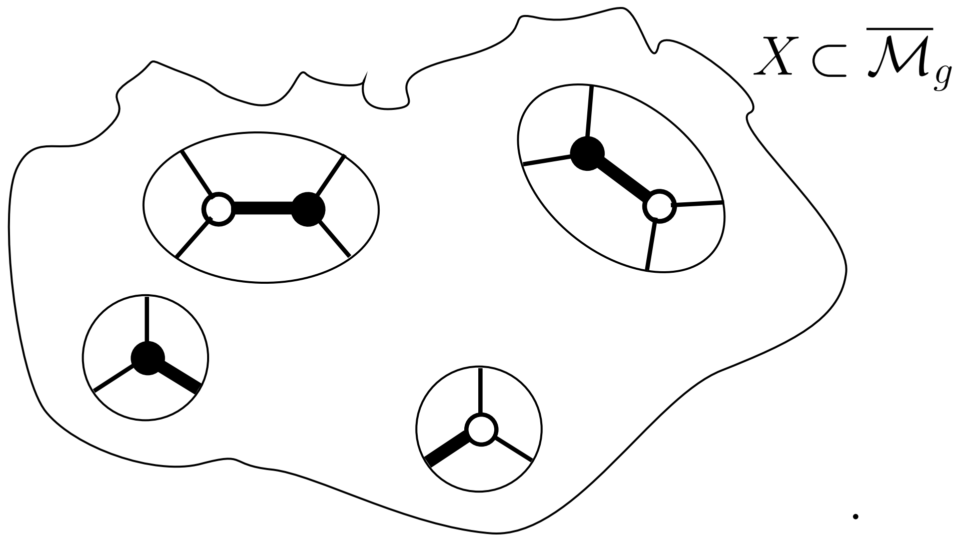
Taking monomer cover



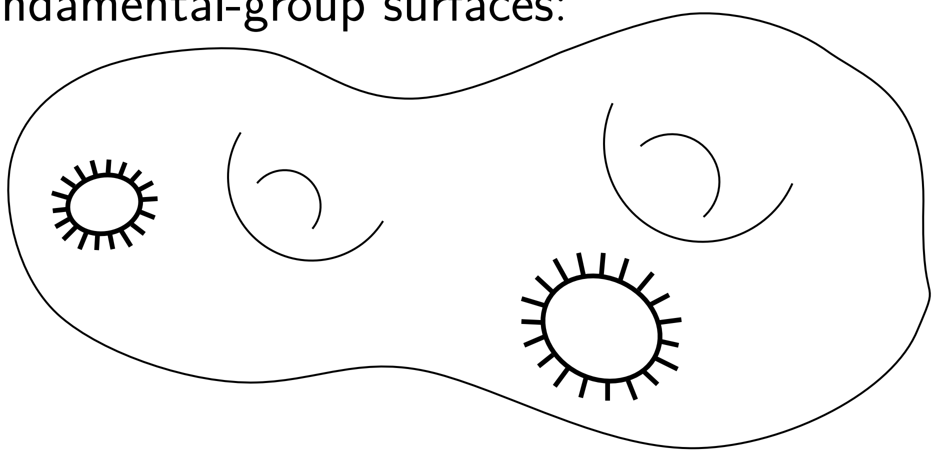
the monomer-monomer observable $M_{b_1 \cdots b_n w_1 \cdots w_n}$ is given by

$$\frac{Z_{X_{b_1 \cdots b_n w_1 \cdots w_n}}}{Z_X}.$$

In particular, adjacent monomers $(b_\ell, w_\ell) \Longrightarrow$ dimer $(i_{b_\ell}\, j_{w_\ell})$, $\forall\, i, j \,|\, \ell \subseteq D$:



Remark. Monomer-monomer observables are a special case of dimer models for nontrivial fundamental-group surfaces:



Remark. $|\{[K]\}| = 2^{2g+2n-1}$, where $2n = |\text{vertices}|$.

1.1.4 Pfaffian polynomials

Theorem. *Orthonormal sequence exists for* 2^{2g} *eigenvalues in multiple by*

$$Z = \frac{1}{2^g} \sum_{[K]} \mathsf{Arf}(q^K_{D_0}) \cdot \varepsilon^K(D_0) \cdot \mathsf{Pf}(X^K) \qquad \Big| \, \mathsf{Arf}(\cdot) \in \{\pm 1\}$$

such that:

$[K]$ = *all equivalence classes of Kasteleyn orientations*, 2^{2g} *in total*

$q^K_{D_0}$ = *quadratic form on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$; *corresponds to Kasteleyn orientation*, *with respect to reference perfect matching* D_0

$$\varepsilon^K(D_0) = (-1)^{\boldsymbol{\sigma}} \, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{2n-1}\sigma_{2n}} \quad \left| \begin{array}{l} \boldsymbol{\sigma} \in \mathit{Aut}(D_0) \subseteq \mathit{Aut}(\mathscr{D}) \\ \sigma \in \boldsymbol{\sigma}\big|_{\mathit{Aut}(D_0)} \subseteq \mathcal{S}_{2n}/(\mathcal{S}_n \times \mathcal{S}_2^n). \end{array} \right.$$

Proof. $\heartsuit$.

Corollary.

(i). For bipartite graphs on $\overline{\mathcal{M}}_g$:

height function =

= section of the non-trivial $\mathbb{Z}$-bundle.

(ii). Fundamental cycles $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is given by:

$$Z\big(\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_g}, \mathcal{H}_{b_1}, \ldots, \mathcal{H}_{b_g}\big) =$$

$$= \sum_{D} \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{i=1}^{g} \exp\Big(\sum_i \mathcal{H}_{a_i} \Delta_{a_i} h + \\ + \sum_i \mathcal{H}_{b_i} \Delta_{b_i} h \Big)$$

where $\Delta_C h$ =

= change in height function along noncontractible cycle C on $\overline{\mathcal{M}}_g$.

Proof. $\heartsuit$.

1.1.5 Computing rank of equivalence classes

The rank $|\sigma\,\big|_{\mathsf{Aut}(D)}| = |\{\widetilde{\sigma}\}| = |\mathscr{D}|$ of an equivalence class of isomorphisms $\mathscr{D}$ for fixed g is a generating function in two-variable $k=2 \mid \omega_1 = 1 = \omega_2$:

$$\sum_{\sigma=\widetilde{\sigma}} \prod_{\ell\in D(\sigma)} \sum_{\xi\,\in\,(\sigma(2\ell-1),\sigma(2\ell))} 1 \quad = \sum_{D(N_1,\dots,N_k)\,|\,(\sum_{\nu=1}^{k} N_\nu)=n} (\pm)\prod_{\nu=1}^{k} \omega_\nu^{N_\nu}$$

$\forall\,\xi$ connecting $\sigma(2\ell-1)$ and $\sigma(2\ell)$; $N_\nu = |\nu$-class dimers$|$.

Derivation I. Let $X \subset \overline{\mathcal{M}}_g =$ planar $M\times N$ square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\cos^2\left(\frac{\pi i}{M+1}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| N = \text{even}$$

$$= |\{\widetilde{\sigma}(X;N,M)\}| \quad \Big| \begin{matrix} M = \text{even} \\ N = \text{odd} \end{matrix}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. $\heartsuit$.

Derivation II. Let $X \subset \overline{\mathcal{M}}_g =$ cylindrical $M \times N$ square grid.

$|\{\widetilde{\sigma}(X; M, N)\}| =$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| N = \text{even}$$

$$= 2^{(\frac{MN}{2} - \frac{M}{2} + 1)} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| \begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. $\heartsuit$.

Derivation III. Let $X \subset \overline{\mathcal{M}}_g =$ toroidal $M \times N$ square grid.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$

$$= 2^{(\frac{MN}{2}-1)} \left(\begin{array}{c} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \sin^2\left(\frac{2\pi j}{N}\right)} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{2\pi i}{M}\right) + \sin^2\left(\frac{\pi(2j-1)}{N}\right)} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \sin^2\left(\frac{\pi(2j-1)}{N}\right)} \end{array} \right) \quad \Bigg| \, N = \text{even}$$

$$= |\{\widetilde{\sigma}(X;\, N, M)\}| \quad \Bigg| \begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big| \, MN = \text{odd.}$$

Show. ♡.

Derivation IV. Let $X \subset \overline{\mathcal{M}}_g =$ planar 6×8 square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$

$$= 16777216\,\Big(\frac{1}{4}+\cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{9})+\cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{7})+\cos^2(\frac{2\pi}{9})\Big)\times$$

$$\times\Big(\cos^2(\frac{\pi}{7})+\sin^2(\frac{\pi}{18})\Big)\Big(\frac{1}{4}+\sin^2(\frac{\pi}{14})\Big)\Big(\cos^2(\frac{\pi}{9})+\sin^2(\frac{\pi}{14})\Big)\times$$

$$\times\Big(\cos^2(\frac{2\pi}{9})+\sin^2(\frac{\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18})+\sin^2(\frac{\pi}{14})\Big)\Big(\frac{1}{4}+\sin^2(\frac{3\pi}{14})\Big)\times$$

$$\times\Big(\cos^2(\frac{\pi}{9})+\sin^2(\frac{3\pi}{14})\Big)\Big(\cos^2(\frac{2\pi}{9})+\sin^2(\frac{3\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18})+\sin^2(\frac{3\pi}{14})\Big).$$

Show. $\heartsuit$.

Derivation V. Let $X \subset \overline{\mathcal{M}}_g =$ cylindrical 6×8 square grid.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$

$$= 5242880 \left(\frac{1}{4} + \cos^2(\frac{\pi}{9})\right)^2 \left(1 + \cos^2(\frac{\pi}{9})\right) \left(\frac{1}{4} + \cos^2(\frac{2\pi}{9})\right)^2 \times$$

$$\times \left(1 + \cos^2(\frac{2\pi}{9})\right) \left(\frac{1}{4} + \sin^2(\frac{\pi}{18})\right)^2 \left(1 + \sin^2(\frac{\pi}{18})\right)$$

Show. ♡.

Derivation VI. Let $X \subset \overline{\mathcal{M}}_g =$ toroidal 6×8 square grid.

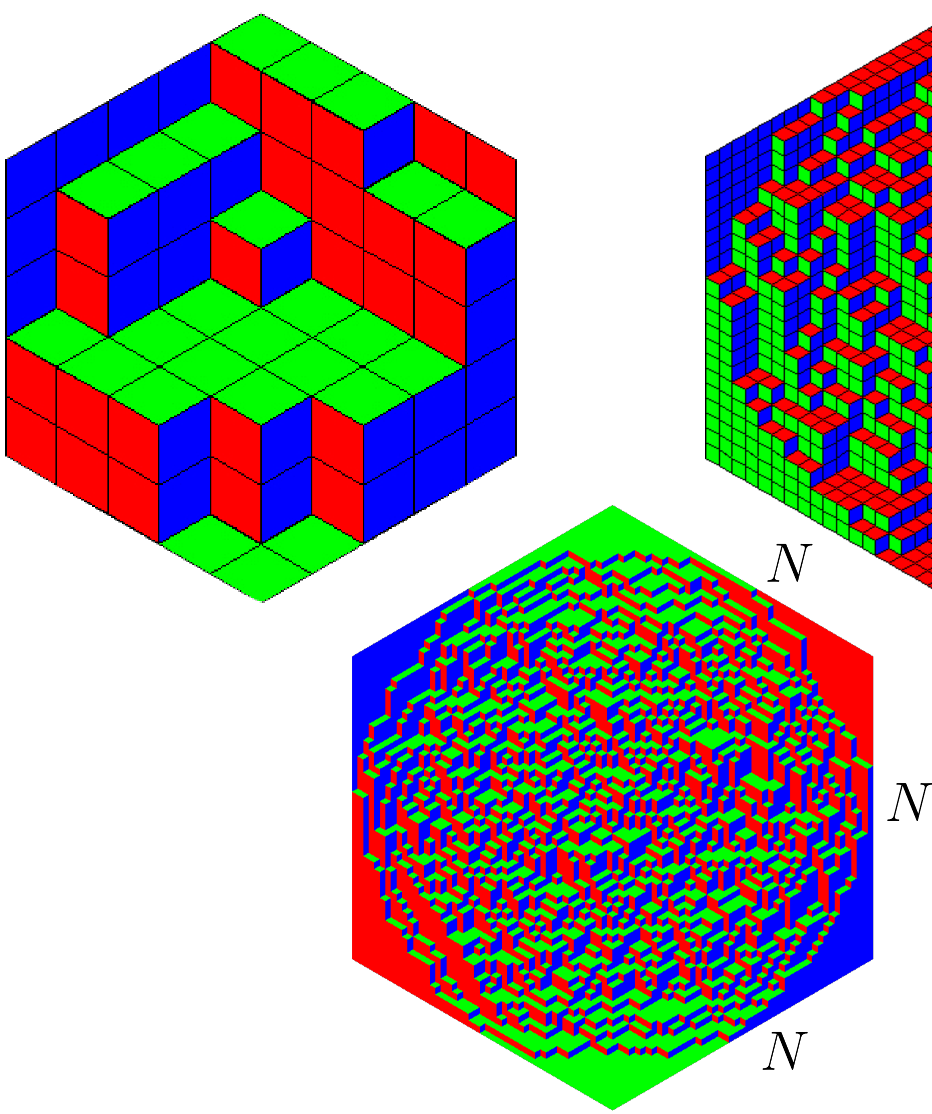
$$|\{\widetilde{\sigma}(X;M,N)\}| =$$

$$= 8388608 \Big[\frac{18225}{131072} + \cos^4(\frac{\pi}{8})\left(\frac{3}{4} + \cos^2(\frac{\pi}{8})\right)^4 \sin^4(\frac{\pi}{8})\left(\frac{3}{4} + \sin^2(\frac{\pi}{8})\right)^4 +$$

$$+ \left(\frac{1}{4} + \cos^2(\frac{\pi}{8})\right)^4 \left(1 + \cos^2(\frac{\pi}{8})\right)^2 \left(\frac{1}{4} + \sin^2(\frac{\pi}{8})\right)^4 \left(1 + \sin^2(\frac{\pi}{8})\right)^2 \Big].$$

Show. ♡.

1.1.6 Limits

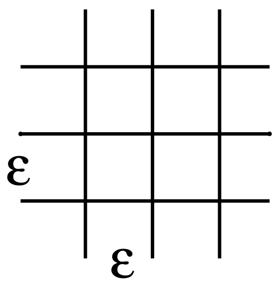


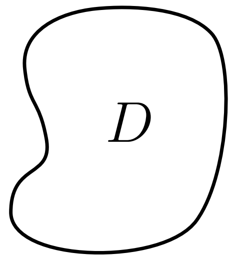
uniform measure

$$\text{Prob}(h) = \frac{1}{|\mathcal{H}_X|}$$

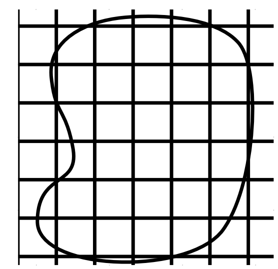
$$N \longrightarrow \infty\,.$$

Theorem (Schur process; Okounkov & R). Let $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \,|\, D \subset \mathbb{R}^2$;





such that: $\begin{cases} \varepsilon \longrightarrow 0, \ |D_\varepsilon| \longrightarrow \infty \\ D_\varepsilon = \varphi_\varepsilon(\mathbb{Z}^2) \cap D \end{cases}$



.

Then, for cube-stack with measure

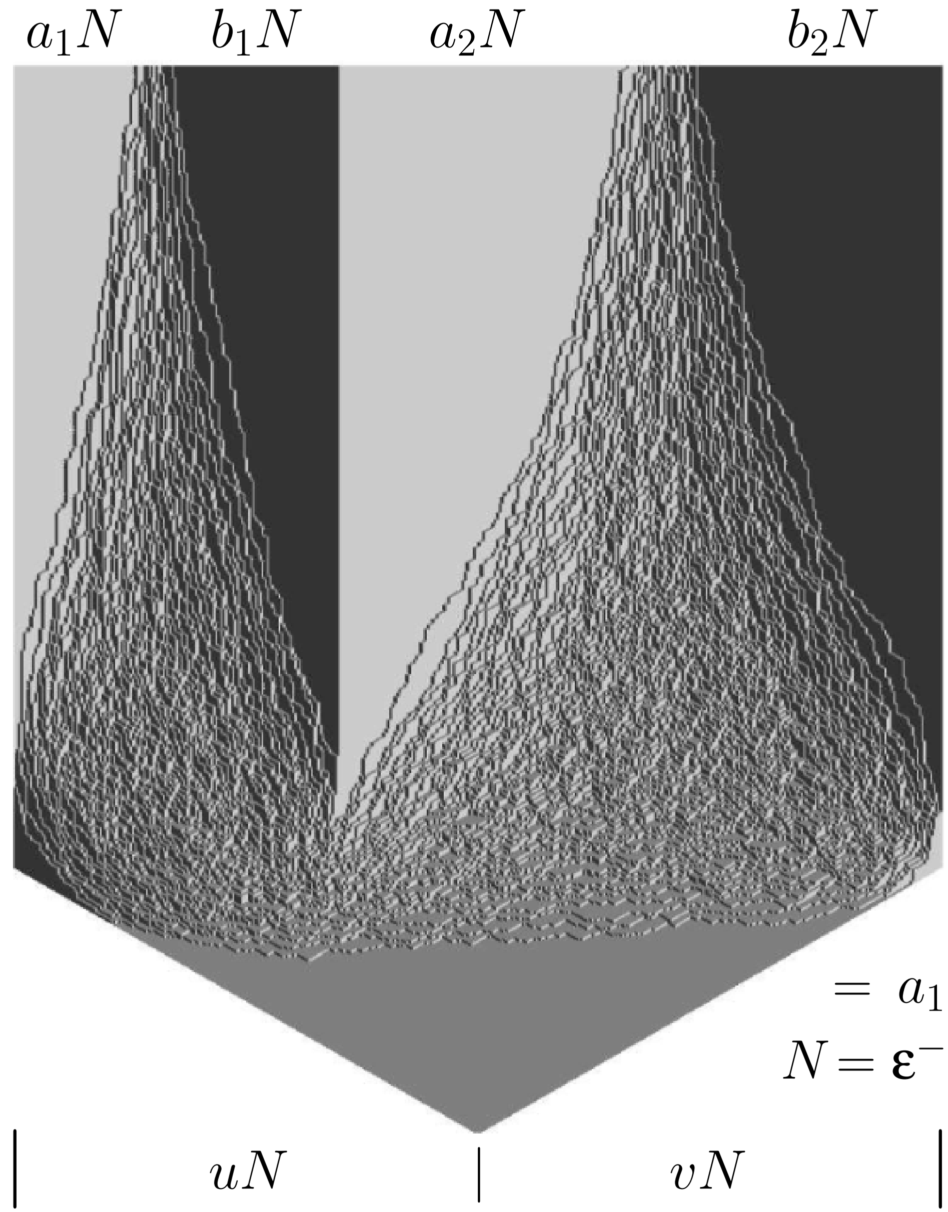
$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \left| \begin{matrix} \pi \in \mathcal{H}_X \\ \pi \cong D, \end{matrix} \right.$$

there is existence of:

$$\textit{Thermodynamic limit } (|D_\varepsilon| \longrightarrow \infty) \ +$$

$$+ \ \textit{Scaling limit } (q = e^{-\varepsilon}, \ \varepsilon \longrightarrow +0).$$

Proof. $\heartsuit$.



where $u+v=$
$= a_1+a_2+b_1+b_2;$
$N=\varepsilon^{-1},\ q=e^{-\varepsilon}.$

1.2 Vertex algebras

Points:

(i) Prove graded kernel convergence for special genus g domain T^*

(ii) Find variational principle in thermodynamic $\ln(\cdot)$ scaling asymptotics

(iii) State conjecture for the Green's function $\langle\cdot\rangle$ in large-deviation

1.2.1 Graded (Grassmann) integral kernels

Pairing $\bigwedge^{\bullet} X^{*} \otimes \bigwedge^{\bullet} X \longrightarrow \mathbb{R} \,|\, \sigma(k)> = (\sigma(1), \ldots, \sigma(k)), \forall\, \sigma(1) > \cdots > \sigma(k),$

$$\langle \varphi(a^{*}), \psi(a) \rangle \stackrel{\text{def}}{=\!=} \varphi_0\, \psi_0 + \sum_{k=1}^{2n} \varphi_k \psi_k + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \varphi_{\sigma(k)\cdots\sigma(1)}\, \psi_{\sigma(1)\cdots\sigma(k)} \; =$$

$$= \; |\psi_0|^2 + \sum_{k=1}^{2n} \int_{\sigma(k)<} |\psi_{\sigma(1)\cdots\sigma(k)}|^2 \, d^{2n} a, \quad \forall\, |\psi|^2 \propto |\varphi|^2 \in \mathbb{R}$$

such that for the dual space, graded basis $a^{*}_{\sigma(k)>}$,

$$\bigwedge^{\bullet} X \ni \psi(a) = \psi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \psi_{\sigma(k)<}\, a_{\sigma(k)<} \;\Big|\; \bigwedge^{k} X \ni \sum \psi_{\sigma(k)<}\, a_{\sigma(k)<}$$

$$\bigwedge^{\bullet} X^{*} \ni \varphi(a^{*}) = \varphi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)>} \varphi_{\sigma(k)>}\, a^{*}_{\sigma(k)>} \;\Big|\; \bigwedge^{k} X^{*} \ni \sum \varphi_{\sigma(k)<}\, a^{*}_{\sigma(k)>}$$

where the dual $\bigwedge^{\bullet} X^{*}$ graded algebra is generated by

$$\left\{ \begin{array}{l} a_0 = 1;\; a_{\sigma(k)<} = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \;\mid\; a_{\sigma(\xi)} \otimes a_{\sigma(\eta)} + a_{\sigma(\eta)} \otimes a_{\sigma(\xi)} = 0; \\ \sigma(k)< \,=\, (\sigma(1), \ldots, \sigma(k)),\; \forall\, 1 \leqslant \sigma(1) \leqslant \cdots \leqslant \sigma(k) \leqslant k = 1, \ldots, 2n \end{array} \right\}.$$

Fixing integrals on $\bigwedge^\bullet V$, $\bigwedge^\bullet V^*$, $\bigwedge^\bullet (V^* \otimes V)$ by choosing

$$a_1, \dots, a_{2n} \in \bigwedge^{2n} V, \quad a_{2n}^*, \dots, a_1^* \in \bigwedge^{2n} V^*$$

and

$$a_{2n}^*, \dots, a_1^*,\ a_1, \dots, a_{2n} \in \bigwedge^{2n} V^* \otimes \bigwedge^{2n} V$$

then

$$\int \bigotimes_{i=1}^{\eta} a_{\sigma(i)}^* \bigotimes_{i=1}^{\eta} a_{\tau(i)}\, da^*\, da = \begin{cases} 0 & , \quad \eta \neq 2n \\ (-1)^{(\sigma + \tau + n(2n-1))} & , \quad \eta = 2n \end{cases}$$

$$\sigma : (\sigma(1), \dots, \sigma(2n)) \longrightarrow (1, \dots, 2n)$$

$$\tau : (\tau(1), \dots, \tau(2n)) \longrightarrow (1, \dots, 2n)\,.$$

Lemma.

$$\big\langle \varphi(a^*), \psi(a) \big\rangle = \int \exp\Big(\sum_i a_i^* a_i \Big)\, \varphi(a^*)\, \psi(a)\, da^*\, da\,.$$

Proof. ♡.

Lemma. *Let* $A\colon V \longrightarrow V$ *by*

$$\psi_A(a) = \sum_{\{i\}_<,\{j\}_<} a_{\{i\}_<} A_{\{i\}_<\{j\}_<} \psi_{\{j\}_<}$$

$$= \psi_0 \oplus A\psi_1 \oplus A^{\otimes 2}\psi_2 \oplus \cdots$$

then

$$\psi_A(b) =$$

$$= \int \exp(-a^* A\, b)\ \exp(-a^* a)\ \psi(a) da^*\, da.$$

Proof. $\heartsuit$.

Lemma.

$$\int \exp(-a^* A\, b)\ \exp(-a^* a)\ \exp(-B^* B\, a)\ da^* da =$$

$$= \exp(-b^* B\, A\, b)\,.$$

Proof. $\heartsuit$.

Remark. Therefore, $\exp(-b^* A\, b) =$ "integral kernel" of A acting on $\bigwedge^{2n} V$.

1.2.2 Vertex operators

(i). The Fermionic Fock space F i.e. $\langle V_m \rangle \in \mathbb{C}^{\,\mathbb{Z}+\frac{1}{2}}$ is given by

$$F = \left\{ V_{m_1} \wedge V_{m_2} \wedge \cdots \;\middle|\; \begin{array}{l} m_i \in \mathbb{Z} + \frac{1}{2} \\ m_{i+1} = m_i - 1 \\ i \gg 1 \end{array} \right\}.$$

(ii). The Clifford algebra is given by

$$Cl_{\mathbb{Z}} = \left\langle \psi_m \, , \, \psi_m^* \right\rangle \;\left|\; \begin{array}{l} m \in \mathbb{Z} + \frac{1}{2} \\ \psi_m \psi_{m'} + \psi_{m'} \psi_m \; = \; \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* \; = \; 0 \\ \psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m \; = \; \delta_{m\,m'} \, . \end{array} \right.$$

(iii). The Clifford algebra acting on the Fock space F:

$$\psi_m \, v_{m_1} \wedge v_{m_2} \wedge \cdots \; = \; v_m \wedge v_{m_1} \wedge v_{m_2} \wedge \cdots$$

$$\psi_m^* \, v_{m_1} \wedge v_{m_2} \wedge \cdots \; = \; \sum_{i=1}^{\infty} (-1)^i \; \delta_{m_i, m} \; v_{m_1} \wedge \cdots \wedge \widehat{v_{m_1}} \wedge \cdots$$

(iv). The Heisenberg algebra is given by

$$\left\langle \alpha_n \right\rangle \quad \left|\begin{array}{l} n \in \mathbb{Z}\backslash\{0\} \\ [\alpha_n, \alpha_{n'}] = -n\,\delta_{n,-n'}\,. \end{array}\right.$$

(v). The Heisenberg algebra acting on the Fock space F:

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m \in \mathbb{Z}+\frac{1}{2}} \psi_{m+n}\, \psi^*_m\,.$$

- As operator in F:

$$\Big[\alpha_n\,,\, \psi_\xi\Big] = \psi_{\xi+n}\,, \quad \Big[\alpha_n\,,\, \psi^*_\xi\Big] = -\psi^*_{\xi-n}\,.$$

(vi). The vertex operators in F are given by

$$X_\pm(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\,\alpha_{\pm n}\Big) \quad \left|\begin{array}{l} \big(X_-(x)v,\, w\big) = \\ \quad = \big(v,\, X_+(x)w\big) = \\ \quad = \big(X_+(x)w,\, v\big). \end{array}\right.$$

(vii). The commutation relations are given by

$$
\begin{aligned}
X_+(x)\,X_-(y) &= (1-x)\ \cdot\ X_-(y)\,X_+(x)\\
X_+(x)\,\psi(z) &= (1-z^{-1}x)^{-1}\ \cdot\ \psi(z)\,X_+(x)\\
X_-(x)\,\psi(z) &= (1-x\,z)^{-1}\ \cdot\ \psi(z)\,X_-(x)\\
X_+(x)\,\psi^*(z) &= (1-z^{-1}x)\ \cdot\ \psi^*(z)\,X_+(x)\\
X_-(x)\,\psi^*(z) &= (1-z\,x)\ \cdot\ \psi^*(z)\,X_-(x).
\end{aligned}
$$

(viii). The eigenvectors are given by

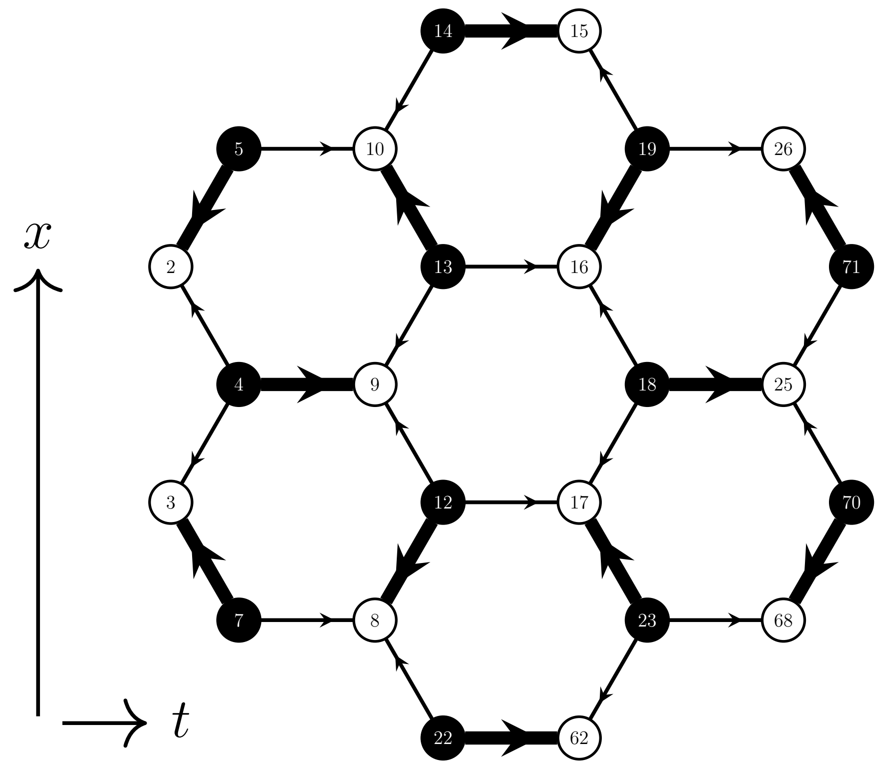
$$
X_-(x)\prod_i \psi^*(w_i)\prod_j \psi^*(z_j)\,V_0^{(n)} =
$$

$$
= \prod_i (1-x\,z_i)^{-1}\prod_j (1-x\,w_j)\prod_i \psi^*(w_i)\prod_j \psi^*(z_j)\,V_0^{(n)}
$$

where $V_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

1.2.3 Fermionic Kasteleyn operators

For the one cube X^* of two-color tiles on bipartite hexagonal lattice X



let the general parameterization for bipartite hexagonal lattice be given by:

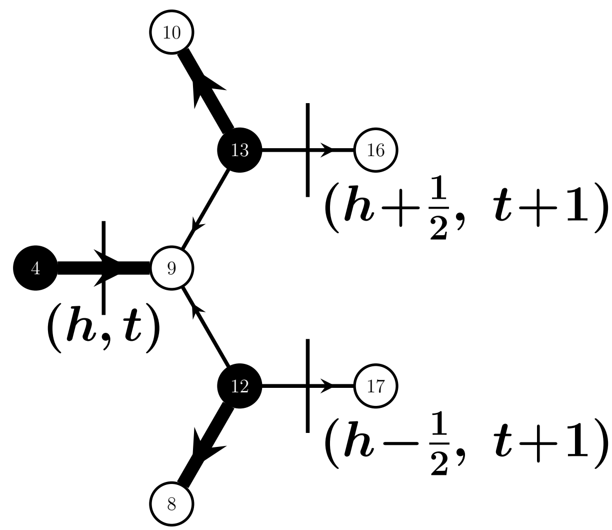
$$b(h,t) = (h,\ t - \tfrac{1}{2})$$

$$w(h,t) = (h,\ t + \tfrac{1}{2}).$$

Kasteleyn matrix by the above-given $b \sim w$ diagram is then given by

$$K(h,t) \;=\; (h,t) \,-\, \big(h{+}\tfrac{1}{2},\, t{+}1\big) \,+\, x_{h,t}\,\big(h{-}\tfrac{1}{2},\, t{+}1\big).$$

Placing Fermions $a^*_{h,\,t}\,,\ a_{h,\,t}$ respectively at $b(h,t)$ and $w(h,t)$:



$$a^*Ka \,=\, \sum_{h,\,t} a^*_{h,t}\, a_{h,t} \,-\, \sum_{h,\,t} a^*_{h+\frac{1}{2},\,t+1}\, a_{h,t} \,+\, \sum_{h,\,t} a^*_{h-\frac{1}{2},\,t+1}\, a_{h,t}\, x_{h,t} \,=$$

$$=\, \sum_{t} \big(a^*_t\, a_t \,+\, a_t\, V\, a^*_{t+1} \,+\, a_t\, V^{-1}\, x_t\, a^*_{t+1}\big).$$

Theorem. *Assuming* $x_{h,t} = x_t,$ *analogous to the notation* $q_{h,t} = q_t,$

$[Diagram]$	*Prob*(π) $\propto \prod_t q_t^{\lvert\pi(t)\rvert}$

the boundary conditions imply

$$Z = \int \exp\bigl(a^* A\, a\bigr)\, da^*\, da \ =$$

$$= \Bigl\langle X_-\bigl(x_{-\frac{1}{2}}\bigr) \cdots X_-\bigl(x_{u_0+\frac{1}{2}}\bigr)\, X_+\bigl(x_{\frac{1}{2}}\bigr) \cdots X_+\bigl(x_{u_1+\frac{1}{2}}\bigr)\, V_0^{(0)}\,, \quad V_0^{(0)} \Bigr\rangle\ .$$

Proof (outline).

$$\int \cdots \exp\bigl(a^*_{t-1}\, a_{t-1}\bigr) \cdot \exp\bigl(a_{t-1}\,(V - V^{-1} X_t)\, a^*_t\bigr) \cdot$$
$$\cdot \exp\bigl(a^*_t\, a_t\bigr) \cdot \exp\bigl(a_t\,(V - V^{-1} X_t)\, a^*_{t+1}\bigr) \cdots \quad =$$

$$= \cdots \underbrace{\overbrace{(V - V^{-1} X_{t-1})}^{-1}}_{X_+(x_t)} \cdot \underbrace{\overbrace{(V - V^{-1} X_t)}^{-1}}_{X_-(x_t)} \cdots$$

where $X_+(x_t)$ and $X_-(x_t)$ each depends on t such that

$$\widetilde{A} = A, \text{ where } V \hookleftarrow \text{ is lifted to } \textstyle\bigwedge^{\frac{\infty}{2}} V \;\Big|\; V = \bigoplus_{m \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}\, v_h$$

under boundary conditions, etc. $\square$

Remark. Direct proof exists combinatorially besides the Kasteleyn method.

Corollary.

$$Z = \prod_{m=\frac{1}{2}}^{u_1 - \frac{1}{2}} \; \prod_{m' = u_0 + \frac{1}{2}}^{-\frac{1}{2}} \bigl(1 - x^-_{m'}\, x^+_m\bigr)^{-1}.$$

Theorem. (Okounkov & R., 2005).

$$\left\langle \boldsymbol{\sigma}_{(h_1 t_1)} \cdots \boldsymbol{\sigma}_{(h_k t_k)} \right\rangle = \det(K((t_i,h_i),(t_j,h_j)))_{1 \leqslant i,\, j \leqslant k}$$

$$K((t_i,h_i),(t_j,h_j)) =$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{|z| < R(t_1)} \int\limits_{|z| < \widetilde{R}(t_2)} \frac{\Phi_-(z,t_1) \ \ \Phi_+(w,t_2)}{\Phi_+(z,t_1) \ \ \Phi_-(w,t_2)} \cdot$$

$$\cdot \ \frac{1}{z-w} \ \cdot \ z^{\left(-\,h_1 - B(t_1) - \frac{1}{2}\right)} \cdot \ w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \ dz\, dw$$

where

$$\left.\begin{matrix} |w| < |z|,\ t_1 \geqslant t_2 \\ |w| > |z|,\ t_1 < t_2 \end{matrix}\ \right| \begin{matrix} R(t) = \min\limits_{m>t}((x_m^+)^{-1}),\ \widetilde{R}(t) = \max\limits_{m<t}(x_m^-),\ B(t) = \frac{|t|}{2} - \frac{|t-u_0|}{2} \\ \Phi_+(z,t) = \prod\limits_{m > \max(t,\frac{1}{2})} (1 - z\, x_m^+),\ \ \Phi_-(z,t) = \prod\limits_{m < \max(t,-\frac{1}{2})} (1 - z^{-1}\, x_m^-). \end{matrix}$$

Proof. ♡.

1.2.4 Thermodynamic limit with scaling

$[Diagram]$

$$\left.\begin{aligned} x_m^+ &= aq^m \\ x_m^- &= a^{-1}q^m \end{aligned}\right\}\text{assumed}$$

corresponding to $\text{Prob}(\pi) \propto q^{|\pi|}$.

Considering limit $\varepsilon \longrightarrow 0$, $q = e^{-\varepsilon}$, $u_1 = \varepsilon^{-1} v_1$, $u_0 = \varepsilon^{-1} v_0$ for fixed v_1, v_0:

$$Z = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - x_m^- x_n^+)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - q^{m-n})^{-1}$$

$$\langle\, |\pi|\, \rangle = q \frac{\partial}{\partial q} \ln Z = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1-e^{\,t-s}}}_{\text{3D volume function}} \, ds\, dt + \cdots$$

where

$$\ln Z = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln(\underbrace{1-e^{-s+t}}_{\text{2D partition function}})\, ds\, dt + \cdots$$

1.2.5 Graded (Grassmann) kernel asymptotics

Consider limit $\varepsilon \longrightarrow 0$ where $t_i = \varepsilon^{-1}\tau_i,\ h_1 = \varepsilon^{-1}\chi_i,$ for fixed $\tau_i,\ \chi_i$:

$[Diagram]$ $(\tau_i,\ \chi_i)$ *in the bulk*

$$K((t_1,h_1),(t_2,h_2)) \longrightarrow$$

$$\longrightarrow \frac{1}{(2\pi i)^2} \int_{C_z} \int_{C_w} \exp\Big(\varepsilon^{-1}(S(z,t_1,\chi_1) - S(z,t_2,\chi_2))\Big) \cdot \cdot\ (zw)^{1/2}\ (z-w)^{-1}\ dz\ dw$$

where

$$S(z,t,\chi) = = -(\chi + \frac{\tau}{2} - u_0)\ \ln Z + \mathsf{Li}_2(ze^{-v_0}) + \mathsf{Li}_2(ze^{-v_1}) - \mathsf{Li}_2(z) - \mathsf{Li}_2(ze^{-\tau})$$

and

$$\mathsf{Li}_2(z) = \int_0^z t^{-1}\ \ln(1-t)\, dt.$$

1.2.6 Critical points

$$\exp\left(\chi + \frac{\tau}{2}\right) = \frac{(1 - ze^{-v_0})(1 - ze^{-v_1})}{(1 - z)(1 - ze^{-\tau})}$$

gives quadratic equation, implying a discriminant for two real solutions or two complex-conjugate solutions, or a zero-discriminant.

$$[Diagram]$$

$$\boxed{\partial_\chi h_0\,(\tau, \chi) = \frac{1}{\pi}\,\mathsf{arg}(z_0)}$$

$$\left\langle \sigma_{(h,\,t)} \right\rangle = K((t, h), (t, h)) \longrightarrow \varepsilon\, \partial_\chi h_0(\tau, \chi)$$

1.2.7 Steepest descent

$$K((t_1,h_1),(t_2,h_2)) \;=\; -\frac{\varepsilon}{2\pi}\cdot\Big(\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\}}{(z_1-w_2)\sqrt{-w_2 S_2''(w_2)}\;\sqrt{z_1 S_1''(z_1)}} \;-$$

$$-\;\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\}}{(z_1-\overline{w}_2)\sqrt{-\overline{w}_2 S_2''(\overline{w}_2)}\;\sqrt{z_1 S_1''(z_1)}} + c.\,c.\Big)\cdot(1+\mathcal{O}(1))$$

That is, for $\mathcal{H}_+ = \{z\in\mathbb{C},\ \mathsf{Im}\,z>0\}\ |\ z_0(\chi,\tau) =$ inner process, such that

$$z_1 \;=\; z_0(\chi_1,\tau_1)$$
$$w_2 \;=\; z_0(\chi,\tau),$$

$$K((t_1,h_1),(t_2,h_2)) \;=$$

$$=\;\frac{\varepsilon}{2\pi}\exp\{\varepsilon^{-1}(\mathsf{Re}(S(z_0(\chi_1,\tau_1)))-\mathsf{Re}(S(z_0(\chi_2,\tau_2)))))\}\cdot$$

$$\cdot\Big(\frac{\exp\{i\,\varepsilon^{-1}(\mathsf{Im}(S'(z_1))-\mathsf{Im}(S(w_2))))\}}{(z_1-w_2)}\;+$$

$$+\;\frac{\exp\{i\,\varepsilon^{-1}(\mathsf{Im}(S'(z_1))-\mathsf{Im}(S(\overline{w}_2))))\}}{(z_1-\overline{w}_2)}\;+\;c.\,c.\Big)\cdot(1+\mathcal{O}(1))\qquad(*).$$

Hence, solution for Kasteleyn-Fermions to free Dirac-Fermions convergence:

$$\frac{1}{\sqrt{\varepsilon}}\,\psi_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + $$
$$+\;\psi_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+\mathcal{O}(1))$$

$$\frac{1}{\sqrt{\varepsilon}}\,\psi^*_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi^*_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + $$
$$+\;\psi^*_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+\mathcal{O}(1))$$

such that

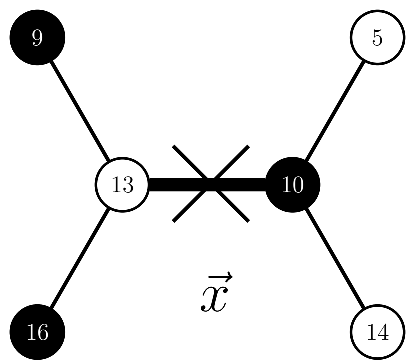
$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\pm(w)) = \frac{1}{z-w}$$

$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\mp(w)) = \mathbb{E}(\psi^*\,\psi^*) = \mathbb{E}(\psi\,\psi) = 0$$

where $\psi^*_\pm(z)$, $\psi_\pm(w)$ are spinors:

$$\psi^*_\pm(z) = \psi^*_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\;, \quad \psi_\pm(z) = \psi_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\;.$$

Remark. The observables are given by:



$$\Big\langle(\sigma_{\vec{x}_1} - \langle\sigma_{\vec{x}_1}\rangle)\,(\sigma_{\vec{x}_2} - \langle\sigma_{\vec{x}_2}\rangle)\Big\rangle = K_{12}K_{21} =$$

$$= \frac{\varepsilon^2}{(2\pi)^2}\left(\frac{\frac{\partial z_1}{\partial x_1}\frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} - \frac{\frac{\partial z_1}{\partial x_1}\frac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} + c.\,c.\right) \times$$

$$\times\ (1 + \mathcal{O}(1)).$$

In particular,

$$\sigma_{\vec{x}_1} - \langle\sigma_{\vec{x}_1}\rangle = \varepsilon\,\partial_x\,\varphi(z_0(\tau, x)) + \cdots \quad \Big|\ \varphi(z) = \text{Gaussian free field on } \mathcal{H}_+$$

such that the Green's function of Dirichlet problem on $\mathcal{H}_+$ is given by

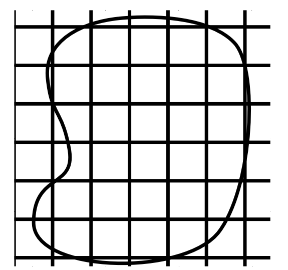
$$\langle\varphi(z)\,\varphi(w)\rangle = \frac{1}{2\pi}\ln\left|\frac{z - w}{z - \overline{w}}\right|$$

and, the Bose-Fermi correspondence is given by

$$\partial_x\varphi = \ :\widetilde{\psi}(z, \overline{z})\ \ \widetilde{\psi}(z, \overline{z}):\ \cdots\ .$$

1.2.8 Scaling limit with Kasteleyn operator

Let $X = D_\varepsilon = \varphi_\varepsilon(L) \cap D$, for arbitrary lattice $L \mid A_G^K =$ difference operator,



where $\varepsilon \longrightarrow 0$ in the asymptotics of the equation for $\mathcal{G}_{x,y}$ given by

$$(A_X^K)_x \cdot \mathcal{G}_{x,y} = \delta_{x,y}$$

Cases.

(i) Hexagonal lattice: Utilizes the weighted as above, for

$$q_t = e^{-\varepsilon f(t)}, \quad t = \frac{\tau}{\varepsilon}, \quad \varepsilon \longrightarrow 0.$$

Theorem. $\mathcal{G}_{x,y} =$ *same as* $(*)$, *with different* $z_0(\tau, x)$.

Proof. $\heartsuit$.

(ii) Periodic lattice: Utilizes variational principle.

1.2.9 Variational principle

(i). For the $N \times M$ torus

$$[Diagram]$$

$$Z(H,V) = \sum_D \prod_\ell \omega(\ell)\, \exp(H\Delta_a h_D + V\Delta_b h_D)$$

$$= \frac{1}{2}\Big\{\mathsf{Pf}\big(A^{K_1}\big) + \mathsf{Pf}\big(A^{K_2}\big) + \mathsf{Pf}\big(A^{K_3}\big) - \mathsf{Pf}\big(A^{K_4}\big)\Big\}$$

where $N, M \longrightarrow \infty$, for fixed $\frac{N}{M}$.

And, $\omega(\ell) = 1 \Longrightarrow$ eigenvalues of Kasteleyn matrices by Fourier transform.

Theorem. (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$\lim_{N,M\longrightarrow\infty} \frac{1}{NM} \ln Z_{NM} = \oint\oint \ln|1+zw|\, \frac{dz}{z}\frac{dw}{w}$$

$$= f(H,V) = \begin{cases} |z| = e^H \\ |w| = e^V. \end{cases}$$

(ii). Taking Legendre transform

$$\boldsymbol{\sigma}(s,\, t) \;=\; \max_{H,V}\left(H_S + V_t - f(H,\, V)\right)$$

then

$$\sum_D 1 \;=\; \sum_D \prod_D w\,(e) \;=\; \exp(MN\,\boldsymbol{\sigma}(s,\, t)\cdot(1+O(1)))$$

where

$$\frac{\Delta_a h_D}{M} = s,\quad \frac{\Delta_b h_D}{N} = t,\quad M, N \to \infty,\quad \frac{N}{M}\text{ fixed.}$$

(iii). For domain

$$[Diagram]$$

$\Delta_a h = sM,\ \Delta_b h = tN.$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\sum_D 1 \;=\; \exp(MN\,\boldsymbol{\sigma}(s,\, t)\cdot(1+\mathcal{O}(1)))$$

with the boundary conditions of height function h_D.

(iv). For domain

$$[Diagram] \qquad M_i \times N_j$$

$$Z_{D_{\boldsymbol{\varepsilon}}} = \sum_{\left\{\substack{\text{values of} \\ \text{height functions} \\ \text{on} \\ \text{boundaries} \\ \text{between rectangles}}\right\}} Z_{\underset{M_i}{\square} N_j}(h_{\text{bound}})$$

$$= \sum_{\{\Delta_x h,\, \Delta_y h\}_{ij}} \exp\left(\sum_{\underset{M_i}{\square} N_j} M_i \, M_j \;\boldsymbol{\sigma}\Big(\frac{\Delta_x h}{M_i}, \frac{\Delta_y h}{N_j}\Big) \right)$$

$$= \exp\left(\boldsymbol{\varepsilon}^{-2} \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\; dx\, dy\; (1 + \mathcal{O}(1))\right)$$

where $h_0 =$ minimizer for

$$S[h] = \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\; dx\, dy.$$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln Z_{D_\varepsilon} = \int_D \sigma(\overrightarrow{\nabla} h_0)\, dx\, dy$$

for $0 < \partial_x h, \partial_y h < 1 \mid h_0 =$ *minimizer*

$h_0\big|_{\partial D} = b,$ *the boundary condition appearing in the limit* $\varepsilon \longrightarrow 0$

$$[Diagram]$$

for height function

$$h = \varepsilon^{-1} h_0 + \varphi = \varepsilon^{-1}(h_0 + \varepsilon\varphi)$$

with respect to $h_0 =$ *limit shape*, *and* $\varphi =$ *distribution (factor).*

1.2.10 Physics way of the continuum process

$$S[h_0 + \varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2} \iint_D a^{ij}(x)\partial_i\varphi\, \partial_j\varphi\, d^2x$$

$$a^{ij}(x) = \partial_i\partial_j\, \varphi(s,t) \begin{cases} s = \partial_1 h_0 \\ t = \partial_2 h_0 \end{cases}$$

such that:

- Partition function equals

$$Z = \exp(\varepsilon^{-2} S(h_0)) \int \exp\left(\frac{1}{2} \iint_D a^{ij}(x)\partial_i\varphi\, \partial_j\varphi\, d^2x\right) D\varphi$$

where $D =$ scalar field with Riemannian metric induced by h_0;

- Correlation equals

$$\langle \varphi(x)\, \varphi(y) \rangle = \mathcal{G}(x,y)$$

where $\mathcal{G} =$ Green's function for $\Delta = \partial_i(a^{ij}\partial_j)$.

Conjecture. $\mathcal{G} =$ *same as obtained by asymptotics of Kasteleyn operators.*

Remark. The conjecture = theorem in certain cases.

Conclusion: continuum process yet

1. How to make such pictures of (i.e. simulate) perfect-matching mixture:
   (i). Monte Carlo for $\exp(\propto 1000^2)$
   (ii). Sampling around most probable region by MCMC
2. How to describe the process and invariant limit analytically:
   (i). Equipartition Pfaffian asymptotics with boundary conditions
   (ii). Variational principle: Minimizer functional in large deviation

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Thank you!