Continuum Limit in Random Partition of Higher Genus

(Based on Discussion with Nicolai Reshetikhin)

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Abstract

For all genus $g \gg$, on bipartite $(\mathbb{Z}^+)^k \subset \overline{\mathcal{M}}_g$ partition, we prove: Polynomial-time mixing correlation $\mathsf{Pf}((X^K)^{-1}) \in \mathbf{Quot}(\mathbb{K}[D])$ of dual spanning-tree projective height distribution; and, the free Dirac Fermion convergence $\Psi = f \cdot (1 + O(1))$ in Grassmann kernel asymptotics, by discriminant steepest descent of thermodynamic scaling limit. We get conjecture: Green's function $G = \langle \cdots \rangle$ of Dirichlet problem for the large-deviation minimizing functional, under variational principle.

Keywords: Continuum-limit, Random-partition, Higher-genus

1. Characterizations

(i) Derive partition Z by bipartite Grassmann integral of height function

(ii) Prove polynomial-time, correlation for all genus-g distribution height

2. Special cases

(i) Formulate the Grassmann kernel in special genus-g domains

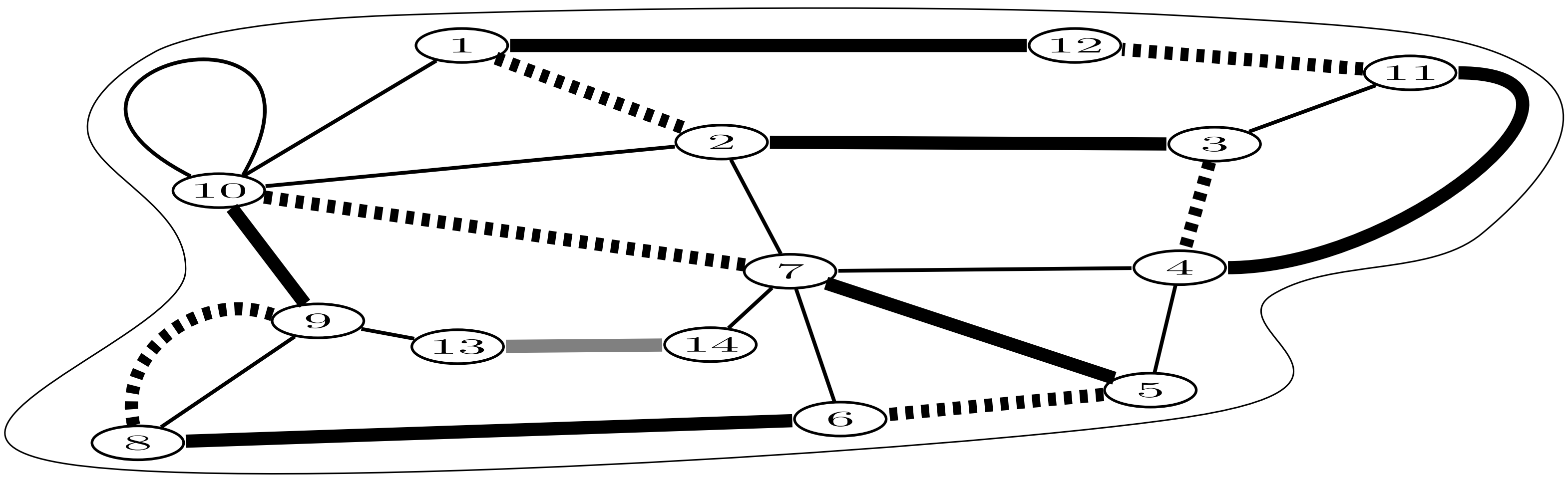
(ii) Find thermodynamic $\ln(\cdot)$ scaling, asymptotics variational-principle

(iii) State conjecture for large deviation functional, Green's function $\langle \cdots \rangle$

1 Characterizations

1.1 Basic definitions and observations

$\forall\, g \gg$, $\exists$ an embedded surface graph $X = (i_\ell, \forall\, i_\ell \neq j_{\ell'} \mid \ell \geqslant \ell';\, i \neq j) \subset \overline{\mathcal{M}}_g$. $X \equiv$ partition $\sigma \in \mathsf{Aut}(\mathscr{D}) \Longleftrightarrow$ perfect-matching $D \mid \mathscr{D} = (D = (\ell)_{\ell \subseteq D}, \forall\, \ell)$, i.e. $\Longleftrightarrow \partial D = (i_\ell = 1, \ldots, 2n) = X$ and $|\bigcap_{\ell \subseteq D}(i_\ell = 1, \ldots, 2n)| = 1$.



Remark. $\overline{\mathcal{M}}_g =$ closed compact orientable. Embedding $=$ cell-complex, i.e. face $\approx$ topological disk, i.e. no hole. Moreover, $\bigcap_{\ell \neq \ell'} (\ell, \ell' \subset D) = \emptyset$, for

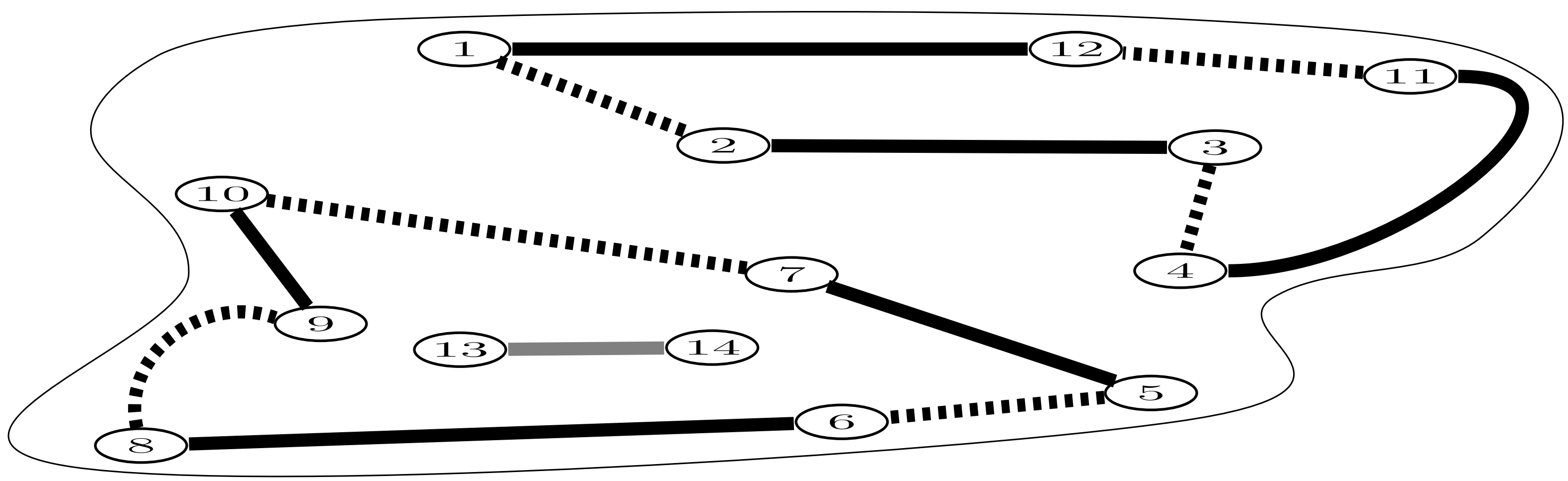
$$\sum_{\ell \neq \ell'} \sigma_D(\ell) = \frac{1}{2}\big|\partial D := X\big| = \frac{|\mathsf{Aut}(\mathscr{D})| \cdot |\{[\sigma]\}|^{-1}}{\exp\{n \ln 2 + \sum_{k=2}^{n-1} \ln k\}} \;\Big|\, \sigma_D(\ell) = \begin{cases} 1 \text{ if } \ell \subseteq D \\ 0 \text{ if } \ell \not\subseteq D. \end{cases}$$

The transition subgraph = symmetric-difference $D_1 \Delta D_2 = D_1 \cup D_2 \setminus D_1 \cap D_2$ $\cong$ homology $\mathcal{H}^1(X;\mathbb{Z}_2) = \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$ class of 1-cycle = union of ordered even-length $\eta = \sum_{C_\alpha} \sigma_{D_1\Delta D_2}(C_\alpha)$ transition cycles, the simple closed paths:

$$C_\alpha = (\ell_{2n_{\alpha-1}+1}, \ldots, \ell_{2n_\alpha}), \quad \forall\, \alpha \in \mathbb{N}^+ \,|\, 1 \leqslant \alpha \leqslant \eta \quad \left| \begin{array}{l} (\ell_{2n_{\alpha-1}+1}, \ell_{2n_{\alpha-1}+3}, \ldots, \ell_{2n_\alpha - 1}) \subseteq D_1 \\ (\ell_{2n_{\alpha-1}+2}, \ell_{2n_{\alpha-1}+4}, \ldots, \ell_{2n_\alpha}) \subseteq D_2 \end{array} \right.$$

where traversing $(i_{2n_{\alpha-1}+1}, \ell_{2n_{\alpha-1}+1}, \ldots, i_{2n_\alpha}, \ell_{2n_\alpha})$

$n_0 = 0.$



Remark. D_1, D_2 are equivalent if $D_1 \Delta D_2 = \emptyset \subset \mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$; $\forall\, D \subset \mathcal{C}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2) =$ 1-chain in cell-complex over $\mathbb{Z}_2 \,\big|\, \partial D \subset \mathcal{C}^0(\overline{\mathcal{M}}_g;\mathbb{Z}_2)$.

Local observable $=$ dimer-dimer correlation (conditional probability)

$$\left\langle \prod_{i=1}^{k} \sigma_D(\ell_i) \right\rangle \stackrel{\text{def}}{=\!=} \text{Prob}(\ell_1 \subseteq D, \ldots, \ell_k \subseteq D) = \mathbb{E}\left[\prod_{i=1}^{k} \sigma_D(\ell_i) \right]$$

$$= \sum_{D \subseteq \mathscr{D}} \prod_{i=1}^{k} \sigma_D(\ell_i) \times \text{Prob}(D) = \frac{\sum\limits_{D \subseteq \mathscr{D}} \prod\limits_{i=1}^{k} \sigma_D(\ell_i) \prod\limits_{\ell \subseteq D} \varepsilon_\ell^K \omega_\ell}{\sum\limits_{D \subseteq \mathscr{D}} \prod\limits_{\ell \subseteq D} \varepsilon_\ell^K \omega_\ell}$$

$$= \frac{1}{Z} \sum_{D \subseteq \mathscr{D} \,|\, \ell_1, \ldots, \ell_k \subseteq D} \varepsilon_D^K \, \omega_D \left| \begin{array}{l} \omega_{(\cdot)} = \prod\limits_{\ell \subseteq (\cdot)} e^{-\frac{\Xi_{(\cdot)}}{\kappa T}} \quad \text{(Boltzmann)} \\ \varepsilon_{(\cdot)}^K = \prod\limits_{\ell \subseteq (\cdot)} \varepsilon_\ell^K = \pm 1, \quad \Xi_{(\cdot)} = \sum\limits_{\ell \subseteq (\cdot)} \Xi_\ell \end{array} \right.$$

positive semi-definite for $Z > 0$, weight $\omega_{(\cdot)} > 0$, and for all vector

$$\mu = (\mu_h = \mathbb{E}[D_h]).$$

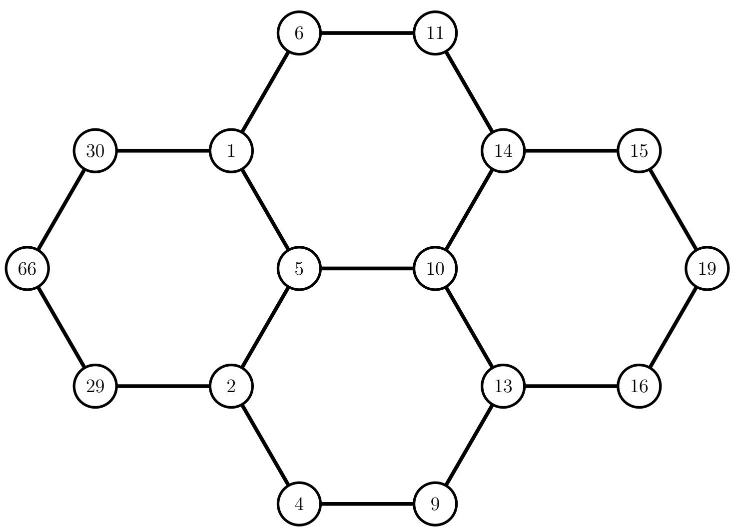
Remark. $\langle \cdots \rangle = 0 \iff \text{Prob}(\cdot) = 0$ if $(\ell_\xi, \ell_\eta) \,|\, \xi \neq \eta$ share common vertex; $\langle \cdots \rangle \big| k = n \Longrightarrow$ normalization; and, $\mathbb{E}\big(\sigma_D(\ell_1)\sigma_D(\ell_2)\big) = \mathbb{E}\big(\sigma_D(\ell_1)\big), \forall\, \ell_1 = \ell_2$.

That is, by $\pm$ signs and $\Xi_{(\cdot)}$, graph $X \overset{\lambda}{\sim}$ dimer σ-finite probability measure $\lambda(\cdot) \in \Xi : E(X) \longrightarrow \mathbb{R} \mid \ell \longmapsto \Xi_\ell \iff X$ is (Boltzmann) weighted:

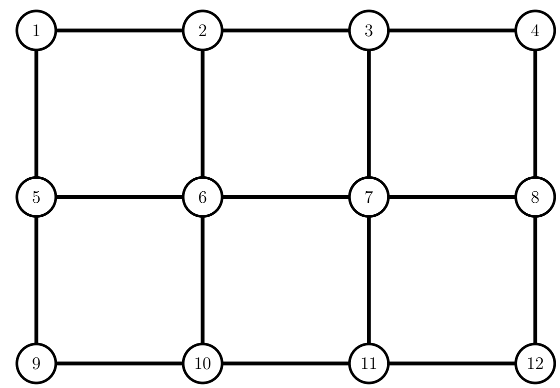
$$\left\langle \prod_{i=1}^{k} \sigma_D(\ell_i) \right\rangle = \frac{\prod\limits_{\ell \subseteq D} \varepsilon_\ell^K \omega_\ell}{\sum\limits_{D \subseteq \mathcal{D}} \prod\limits_{\ell \subseteq D} \varepsilon_\ell^K \omega_\ell} = \frac{\varepsilon_D^K \omega_D}{Z} = \mathsf{Prob}(D) \Bigg| \omega_{(\cdot)} = e^{-\frac{\Xi_{(\cdot)}}{\kappa T}},\ \varepsilon_{(\cdot)}^K = \prod_{\ell \subseteq (\cdot)} \varepsilon_\ell^K$$

$$= \frac{1}{Z}\ \varepsilon_D^K \exp\left(-\frac{\Xi_D}{\kappa T}\right) \Bigg| Z = \sum_{D \subseteq \mathcal{D}} \varepsilon_D^K \exp\left(-\frac{\Xi_D}{\kappa T}\right),\ \Xi_{(\cdot)} = \sum_{\ell \subseteq (\cdot)} \Xi_\ell$$

where $Z =$ strict-sense positive, continuous partition function on objects:



- (regular) Hexagonal grid domains.

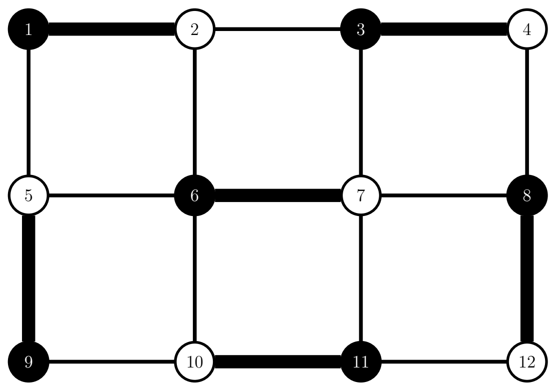


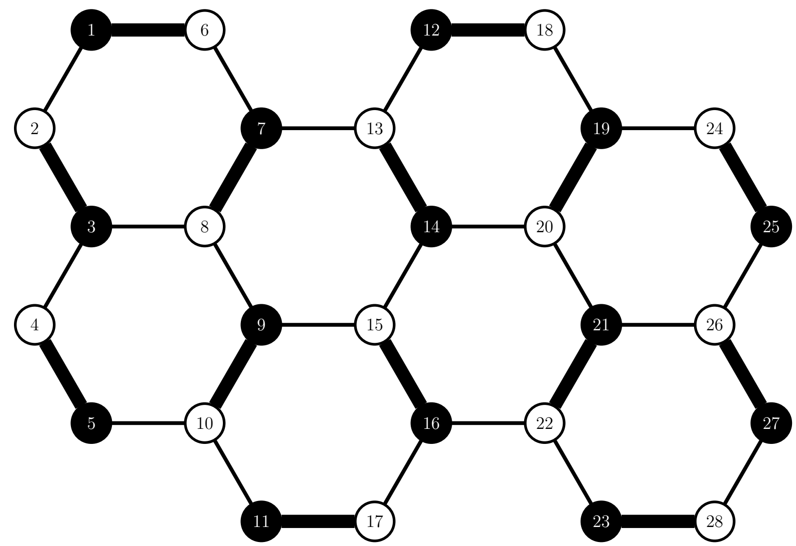
- Square grid domains.

Remark. Bipartite $\Longrightarrow$ no adjacent-black or -white vertices, for well-defined path cartesian product: $M\times N$ vertices ($(M-1)\times(N-1)$ edges), $2n=MN$ quadratic lattice

$$V(X) \;=\; V_{\bullet}(X) \sqcup V_{\circ}(X).$$

Instance.





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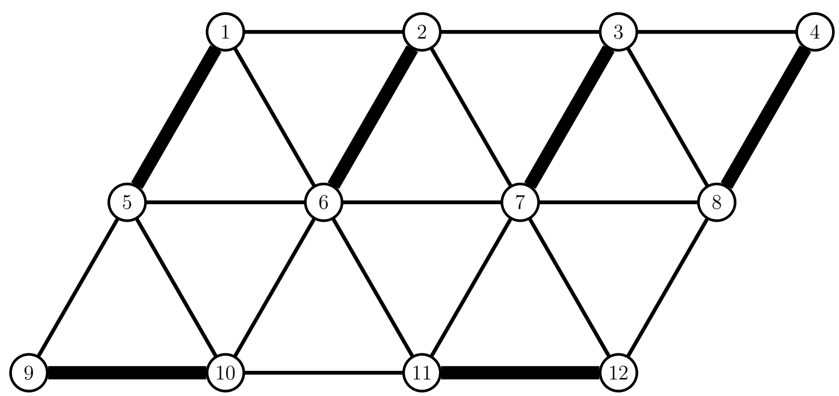
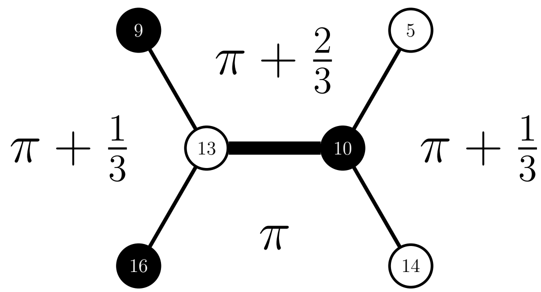
Non-instance.

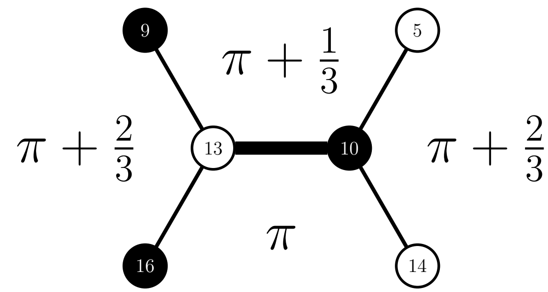


$\left(\begin{array}{l}\textit{no bipartite structure}\\ \textit{on triangular grids}\end{array}\right).$

The space of height functions is parameterized by orthogonal projections:

$$\mathcal{H}_X \stackrel{\text{def}}{=\!=} \{\pi : \text{ faces}(X) \longrightarrow \mathbb{Z}\}$$





with respect to reference face f_0 boundary-normalization $\pi(f_0) = 0$, for all:

$X \subset \mathbb{R}^2 =$ bipartite, hexagonal embedding of

$$\boldsymbol{Dimers} \longleftarrow\!\!\! \dashv \boldsymbol{Discrete\ surfaces}.$$

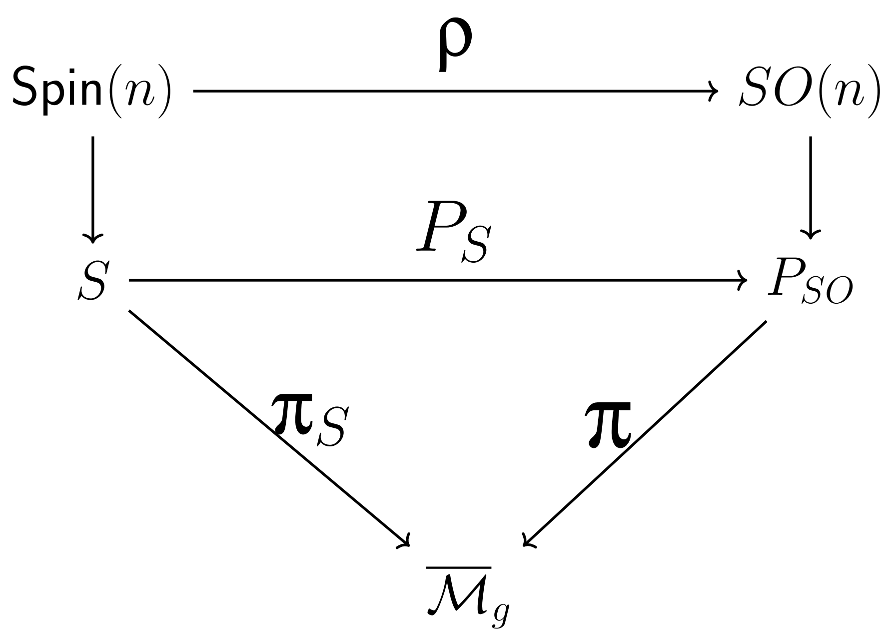
Proposition (height-face).

(i) $\pi_D\big|_{\partial X} = \pi_D$ *restricted to boundary faces* ∂X *is independent of* D.

(ii) $\pi_{D_1 D_2} = \pi_{D_1} - \pi_{D_2}$.

Proof. $\heartsuit$.

Spin structure $\equiv$ spin (spinor) bundle $=$ equivariant 2-fold cover S of the oriented orthonormal frame (principal) bundle $\pi : P_{SO} \longrightarrow \overline{\mathcal{M}}_g =$ orientable surface, for double covering $\rho :$ Spin$(n) \longrightarrow SO(n)$, where structure group Spin(n) for spinor space Δ_n is double cover of orthogonal group $SO(n)$, and the following diagram is commutative:



$$\pi_S = \pi \circ P_S$$

$$P_S(p\,q) = P_S(p)\;\rho(q)$$

$$p \in S$$

$$q \in \text{Spin}(n).$$

Remark. Complex vector bundle $\equiv$ bundle of spinors $\pi_S : S \longrightarrow \overline{\mathcal{M}}_g$ with respect to principal bundle π and spin representation of Spin(n) on Δ_n.

1.2 What is known

1.2.1 Number of $\pm$ Pfaffians

Kasteleyn (1963). For $g = 0$, $Z = \pm$ Pfaffian of Kasteleyn matrix.

Kasteleyn (1963). For $g = 1$, $Z =$ linear in 4 Pfaffians; 3 "$+$", 1 "$-$".

Kasteleyn (1963). For $g > 1$, $Z =$ conjecture: Mysterious 2^{2g} Pfaffians; project was not finished, at least, not published.

1.2.2 Combinatorics of $\{\pm\}$

Gallucio & Loebl (1999). $Z := \pm 1$ for compact orientable surface.

Tesla (2000). $Z := (\sqrt{-1}, \pm 1)$ for non-orientable surface; $\cong$ |Paffians|.

Cimazoni & R. (2004 - 2005). $Z := \pm 1$ by spin-structures.

Cimasoni (2006). $Z := \sqrt{-1}$ by the orientable double-cover pin-minus structure; G. Tesla (2000) topological model; $\sqrt{-1} \cong$ spin structure's ± 1.

1.2.3 Pfaffian asymptotics

R. et al. (2000s). By height functions $h(\mathcal{F})$ with face-weights $q_{\mathcal{F}}$,

$$Z \;=\; \sum_{D}\,\prod_{\ell\,\in\,D} \omega_\ell \;=\; \text{Const.} \times \sum_{h}\,\prod_{\mathcal{F}} q_{\mathcal{F}}^{h(\mathcal{F})}\,.$$

With entropy (as $|X| \longrightarrow \infty,\;\, q_{\mathcal{F}} \longrightarrow 1$) given by the Gaussian field theory (Seiberg-Witten conjecture) in the path integral of scaling limit,

$$Z \;=\; \int \exp\left\{ -\,\frac{1}{2}\left(\int\limits_{\overline{\mathcal{M}}_g} (\partial\Phi)^2\, d^2x \;+\; \int\limits_{\overline{\mathcal{M}}_g} \lambda(x)\,\Phi(x) \right)\right\}$$

where R.H.S linear multiple $\lambda(x)\,\Phi(x)$ is given in terms of $q_{\mathcal{F}}^{h(\mathcal{F})}$ by:

$$q_x \;=\; \ell^{-\varepsilon\;\lambda(x)} \qquad \Big|\, \varepsilon = \text{lattice step};\;\, \lambda = \text{logarithmic scale, as } \varepsilon \longrightarrow 0.$$

That is, by Alvarez-Gaumé, Moore, Nelson & Vafa (1986), studying Fermi and Bose partition correspondence on Riemann surfaces,

$$\text{R.H.S.} \;\sim\; \sum_{\xi\,\in\,S(\overline{\mathcal{M}}_g)} \text{Arf}(\xi) \;\times\; |\Theta(z \mid \xi)|^2 \qquad \Big|\, z = \text{determined by } \omega\text{'s}.$$

Remark. Natural conjecture is born: In large thermodynamic-scaling limit, as correlation decay linearly to critical ω's, asymptotics of $Z(\text{Pfaffians})$ is

$$e^{\textbf{Volume}} \times \textbf{the free energy}$$

where the next leading term is sum of theta functions, and square of each theta function is next leading asymptotics of each Pfaffian, respectively.

The conjecture was confirmed in:

(i) **Ferdinand (1967).** *For square-grid torus.*

(ii) **Costa-Santos & McCoy (2002).** *For $g \geqslant 2$, it is numerically*

$$\text{Arf}(\xi) \times |\Theta(z \mid \xi)|^2 .$$

That is, the conjecture works, however, still a conjecture i.e. no proof yet.

Remark. (i) Although correlation is given by logarithmic ω derivative, the entropy model is the preferred sophistication for fixed (not varying) genus.

(ii) $Z =$ glueable (summable) on boundary spins for surfaces with boundary.

(iii) "Higher" spin structure is unknown, perhaps a para-polynomial theory.

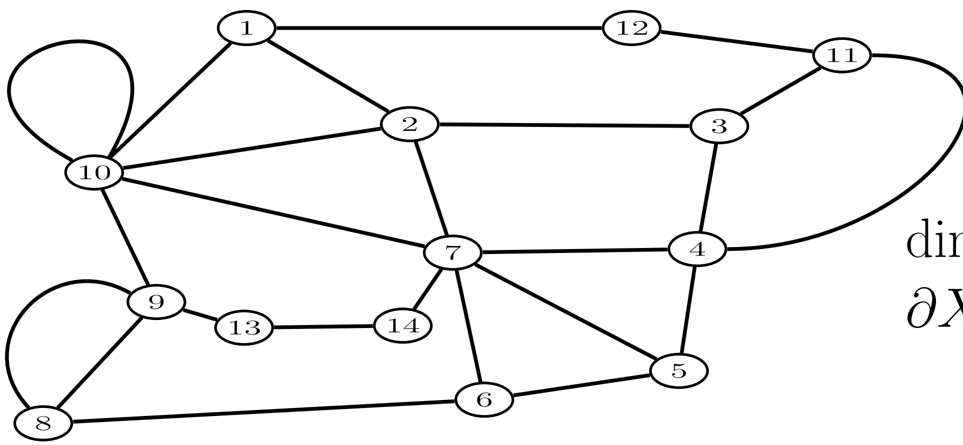
1.3 Combinatorial equivalence

Lemma (coloring-isomorphism). *Given space of tilings, resp. dimers,*

$$\boldsymbol{family}\ (\boldsymbol{Dimers}) \longleftrightarrow \boldsymbol{family}\ (\boldsymbol{Tilings}).$$

Proof. Let $X \subset \mathbb{R}^2$ = closed connected, planar (non-intersecting edges); the union of all X spanning trees defines:

(i) For 2D cell complex X:
0-cells, 1-cells, 2-cells = vertices, edges, faces, resp.

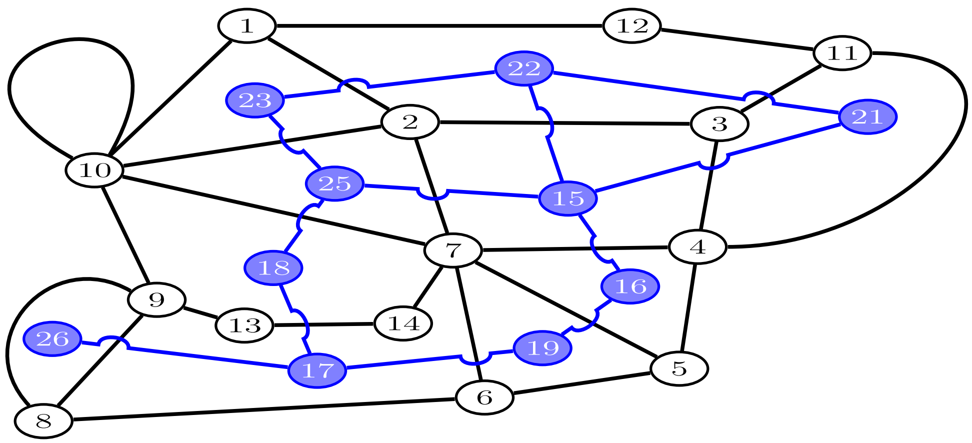


Disjoint interiors.

$\dim(\partial X^{(k)}) = (k-1) \bmod 2$
$\partial X^{(k)}$ = *boundary of two k-cells*,
$k = 0, 1, 2.$

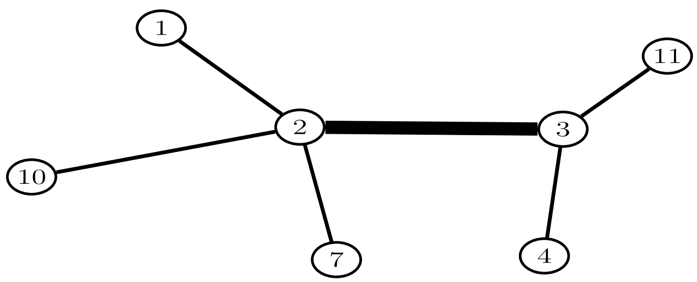
Remark. $X \subset \overline{\mathcal{M}}_g$ = embedding i.e. generally, 1-skeleton CW-complex (resp. cell-decomposition of orientable, closed connected genus g surface).

(ii) For dual cell complex X^* (the union of all X^* spanning trees), 0-cells, 1-cells, 2-cells $=$ resp. "*centers*" of 2-cells, 1-cells, 0-cells of X:

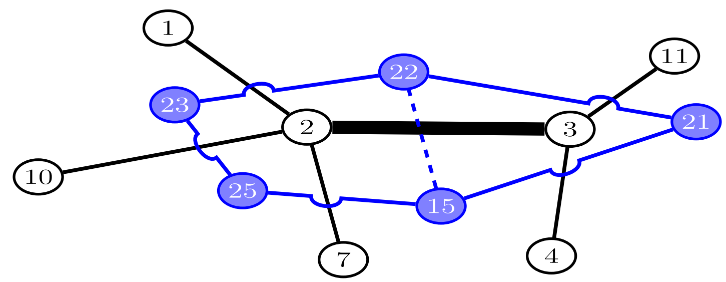


$X^* =$ dual cell complex to X.

(iii) For a dimer on X:



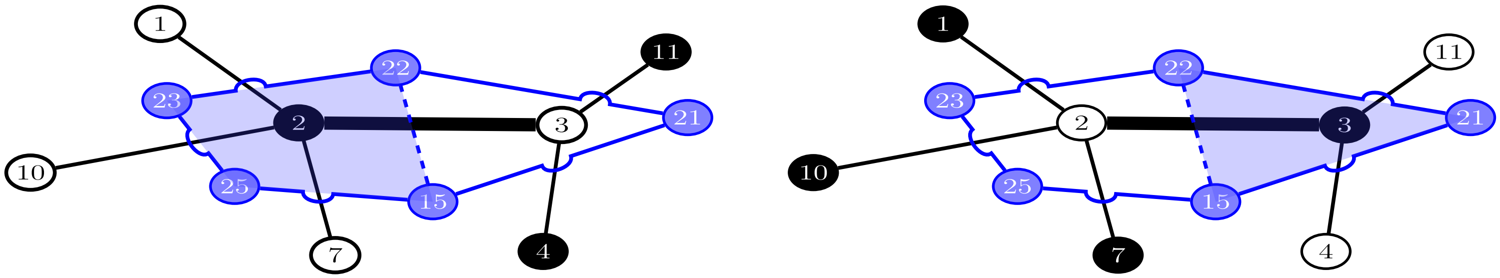
Unique pair of 2-cells on X^* share:



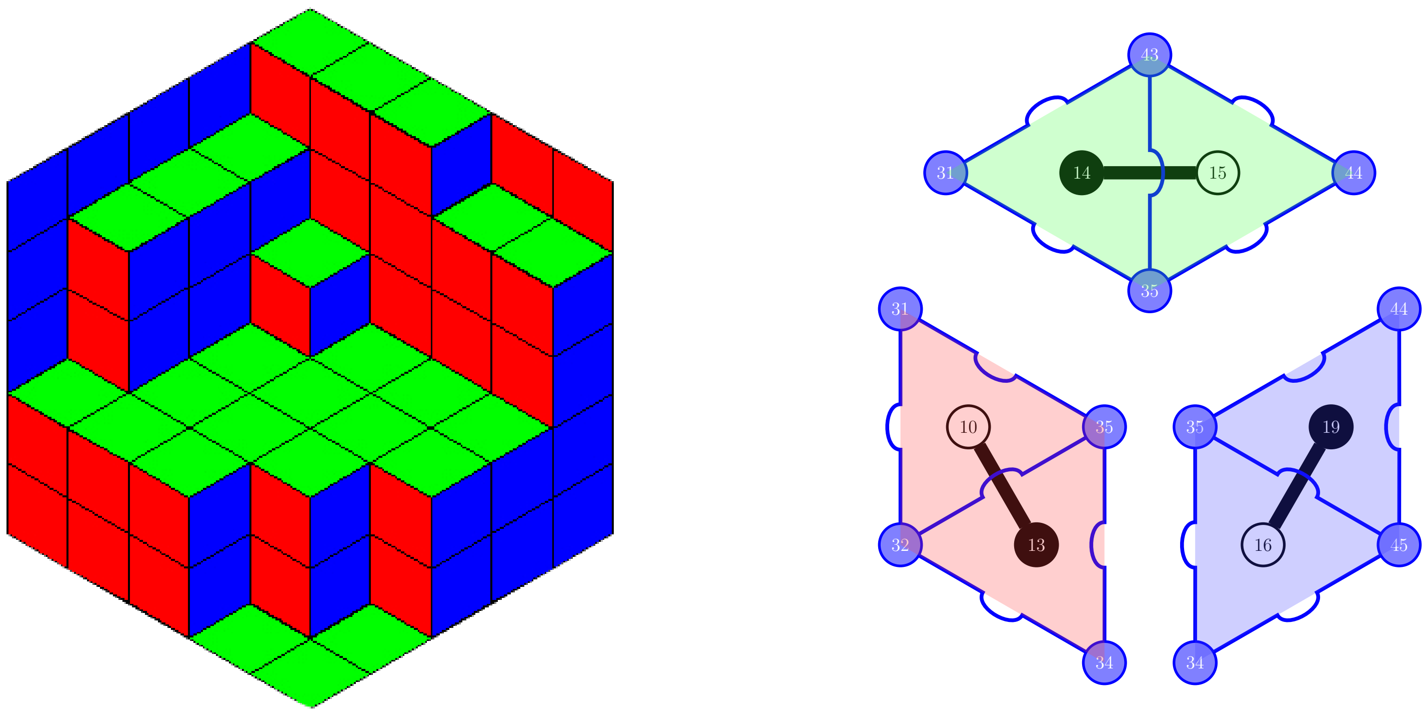
(iv) Therefore, the global bijection:

$$(\text{Dimers on } X) \longleftrightarrow \begin{pmatrix}\text{Tilings of } X^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells}\end{pmatrix}. \qquad \square$$

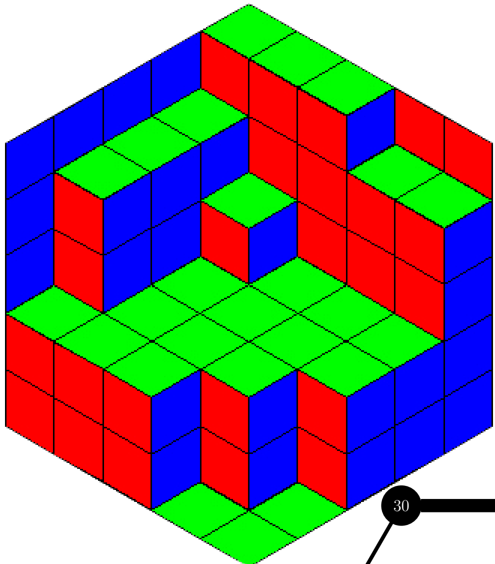
Remark. On bipartite graph:

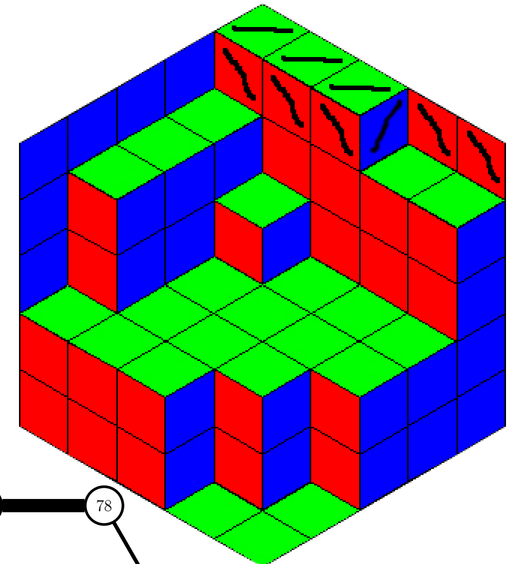


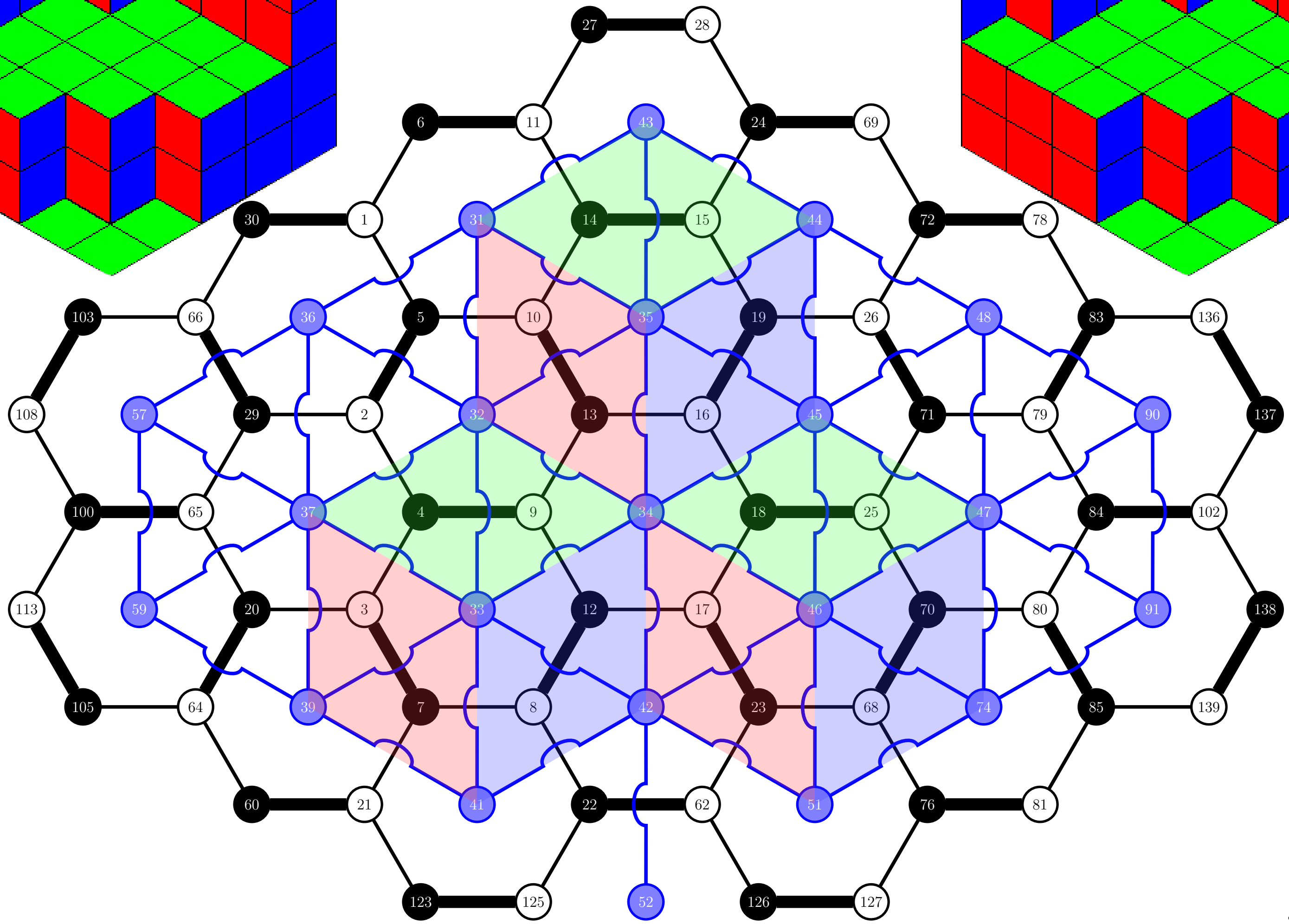
(two-color tiles, shown-above, are admissible)



Cubes: 3D boxes by 2D rhombus-tiling projection







Theorem. *Dimer probability* $\text{Prob}(D) = (1/Z)\prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi_D(\mathcal{F})}$ *equals measure*

$$\text{Prob}(\pi) = \frac{1}{Z}\prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi(\mathcal{F})} \quad \Big| Z = \sum_{\pi \in \mathcal{H}_X} \prod_{\mathcal{F}} q_{\mathcal{F}}^{\pi(\mathcal{F})}, \; q_{\mathcal{F}} = \prod_{\ell \in \partial\mathcal{F}} \boldsymbol{\omega}_\ell^{\boldsymbol{\varepsilon}_\ell^K}.$$

Proof. The combinatorial equivalence i.e. coloring-isomorphism

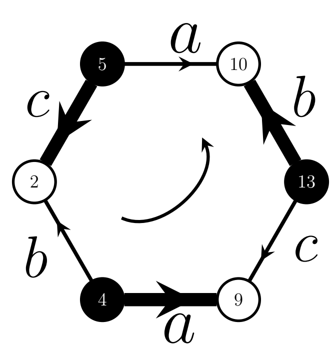
$$\Big\{\textit{Dimers on } X\Big\} \underset{\textit{bijection}}{\cong} \Big\{\textit{height functions}\Big\}$$

with the height-face proposition $\Longrightarrow$ $\text{Prob}(D) = \text{Prob}(\pi)$. □

Remark. $\text{Prob}(D) =$ "gauge" invariant measure: $\boldsymbol{\omega}_\ell \longmapsto s(\ell_+)\ \boldsymbol{\omega}_\ell\ s(\ell_-)$. Furthermore, $q_{\mathcal{F}} =$ invariant ("essential" parameters).

Cases.

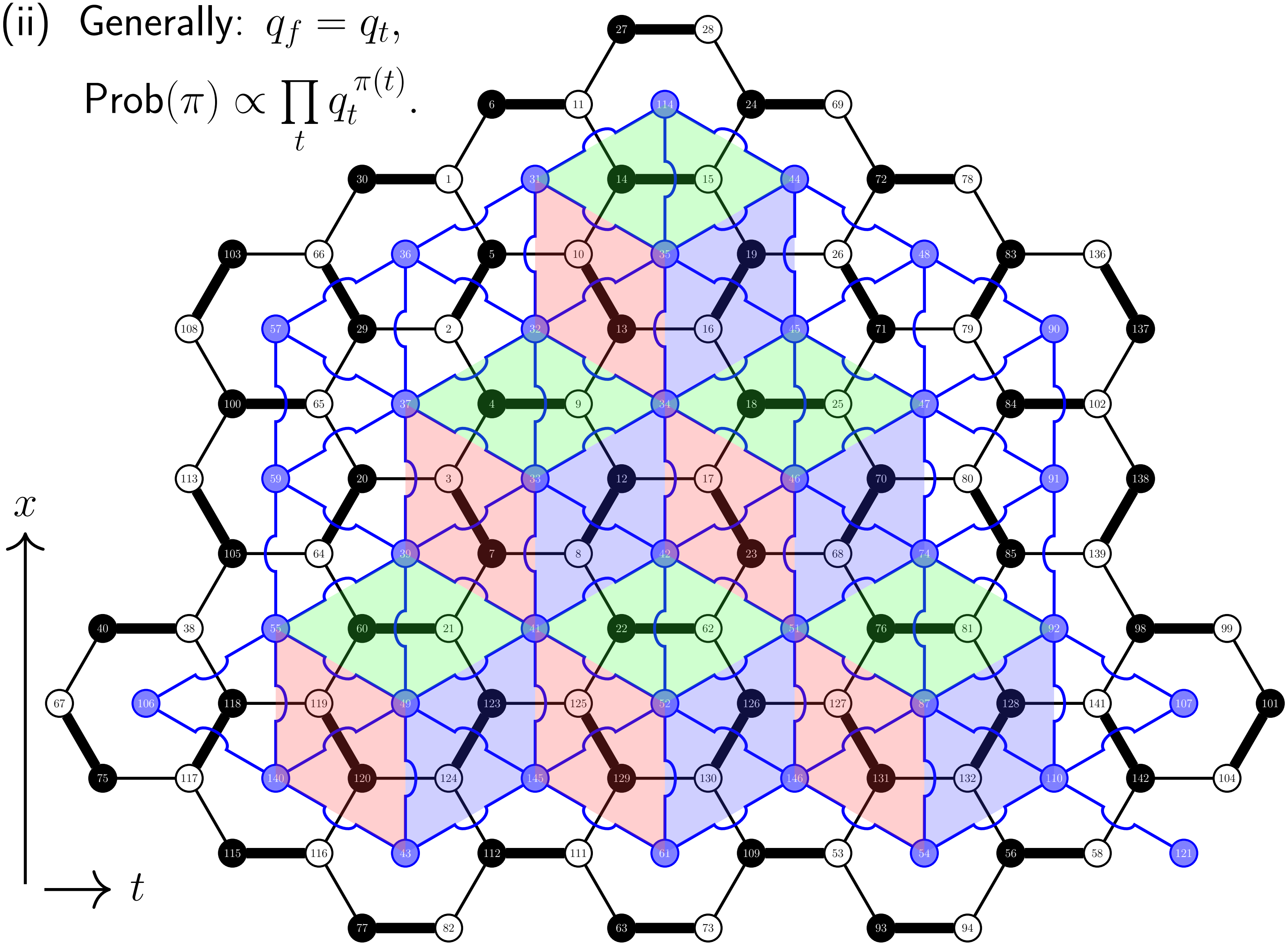
(i) Uniform distribution:



$$q = a^{-1} b\, c^{-1} a\, b^{-1} c = 1.$$

(ii) Generally: $q_f = q_t$,

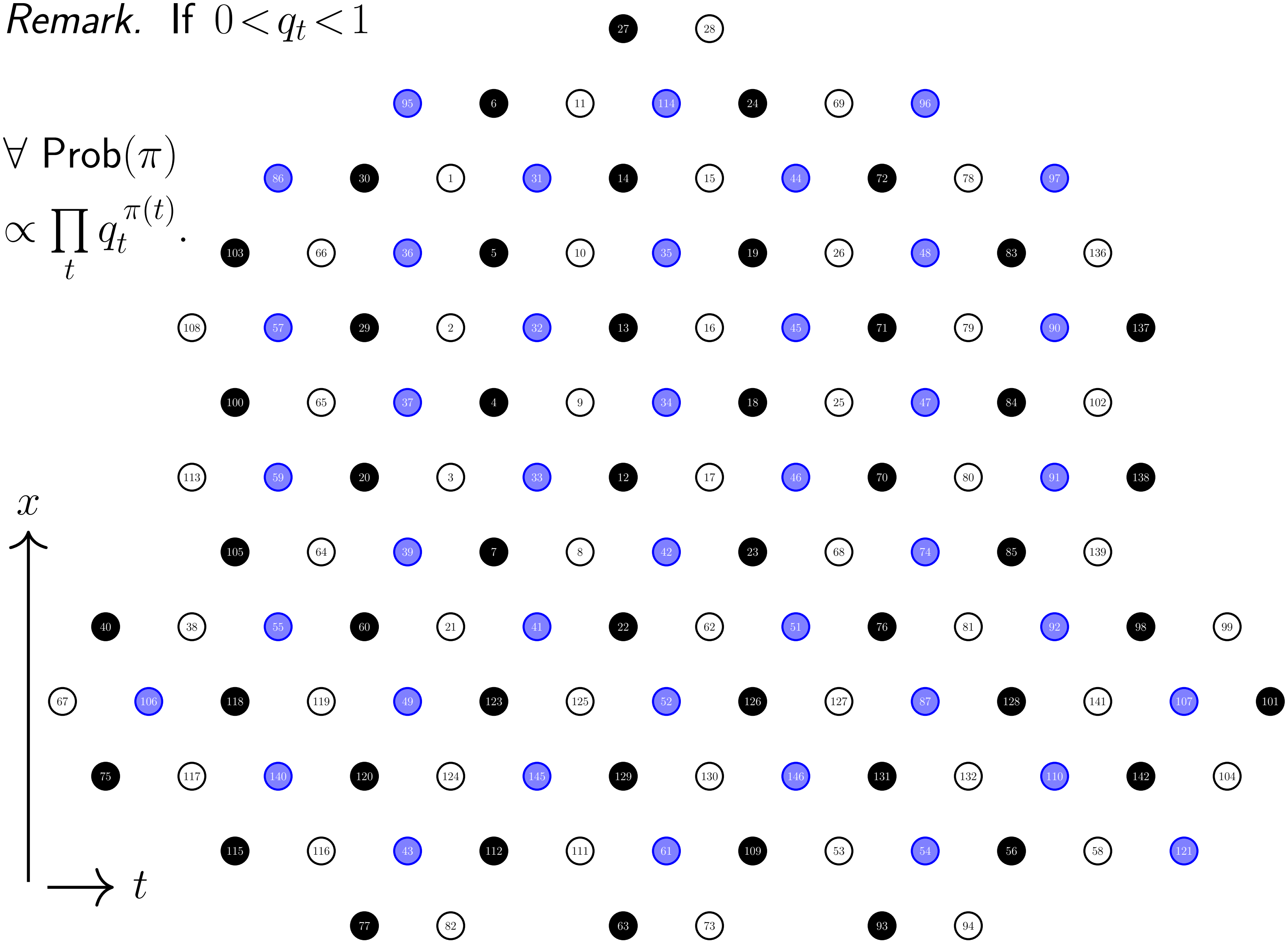
$$\text{Prob}(\pi) \propto \prod_t q_t^{\pi(t)}.$$



Remark. If $0 < q_t < 1$

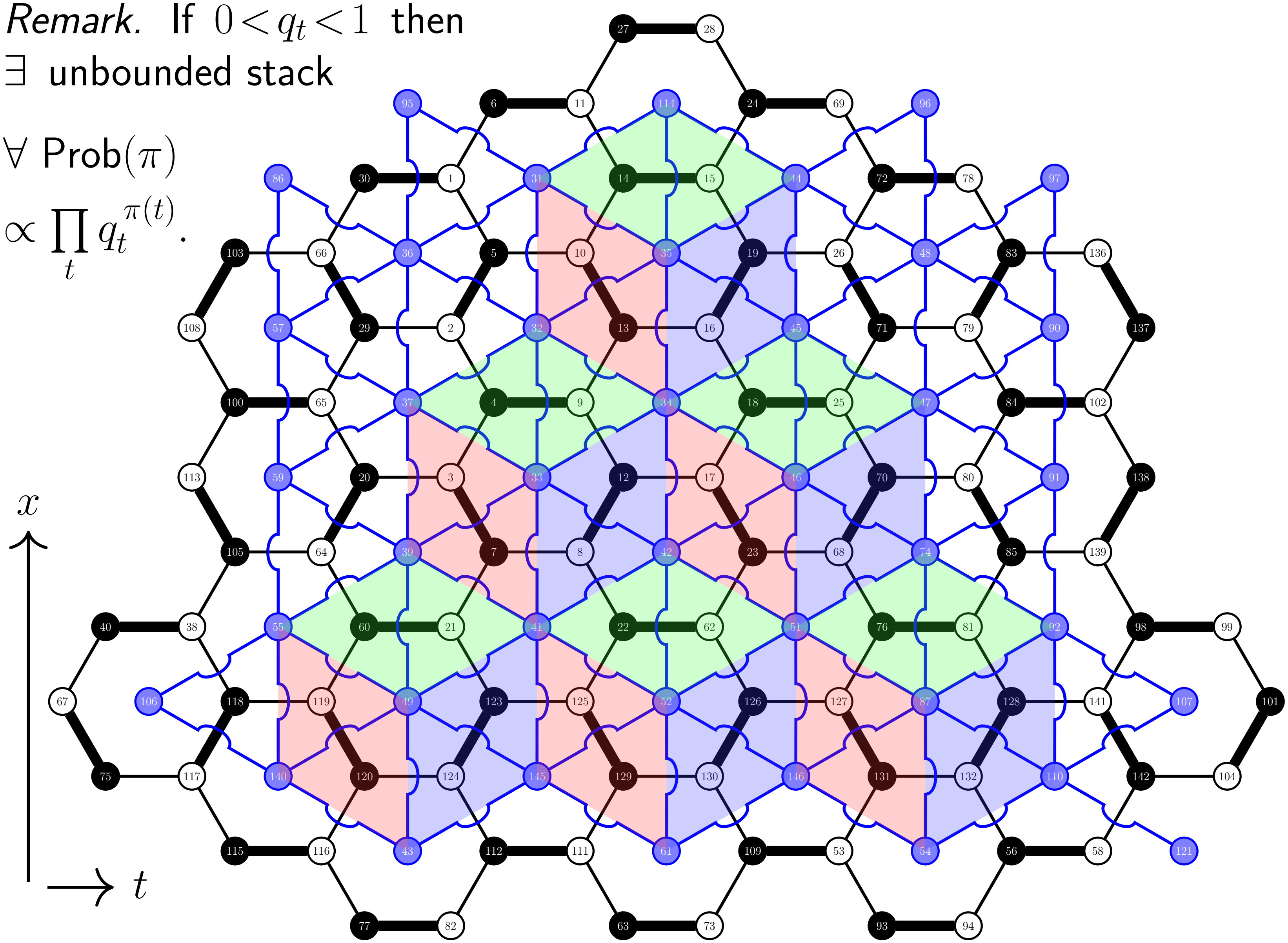
$\forall$ Prob(π)

$\propto \prod_t q_t^{\pi(t)}$.



Remark. If $0 < q_t < 1$ then
$\exists$ unbounded stack

$\forall$ $\mathrm{Prob}(\pi)$
$\propto \prod_t q_t^{\pi(t)}$.



Lemma (perfect-matching). $\forall\,\mathscr{D}$, *equivalences' rank* $|\{\widetilde{\sigma}\}|=|\{[\sigma]\}|$,

(i) $|\{\widetilde{\sigma}\}| \leqslant \sqrt{(2n)!\cdot 2^{-((1/\varepsilon)\ mod\ d(n))}\cdot e^{\ln(g(n)\cdot f(n))}}$, $\varepsilon \longrightarrow 0,\ n \longrightarrow \infty$.

(ii) $\min(\deg(X))\geqslant(2n-1),\ \ \forall\,\langle g(n), f(n)\rangle \in \mathbb{R}$,

$$\Longleftrightarrow g(n)\cdot n!\cdot e^{\ln f(n)} = (2n-1)!! = (2n-1)\cdots 3\cdot 1 = \prod_{k=0}^{n-1} 2k+1.$$

Proof. Albeit perfect-matching rank $|\mathscr{D}|$, $\mathcal{S}_{2n}\supseteq \mathrm{Aut}(\mathscr{D})=\{\widetilde{\sigma}\}\times(\mathcal{S}_n\times\mathcal{S}_2^n)$:

$$\sum_{D(N_1,\ldots,N_k)\,|\,(\sum_{\nu=1}^{k} N_\nu)=n} 1\ \Bigg|\, X\longleftarrow X : \widetilde{\sigma}=\begin{cases}\sigma(1,\ldots,2n)=(\sigma(1),\ldots,\sigma(2n))\\ \sigma(2\ell) > \sigma(2\ell-1),\ \ \forall\,\ell\subseteq D\subseteq\mathscr{D}\end{cases}$$

$\forall\, g, f$, such that $N_\nu=|\text{edges}|$, $\forall\,\nu$-class; $|\{\widetilde{\sigma}\}| = \Big|\{[\sigma]\}\subseteq\mathcal{S}_{2n}\big/(\mathcal{S}_n\times\mathcal{S}_2^n)\Big|$:

$\mathcal{S}_n = \{\,(\sigma(1),\ldots,\sigma(2n)),\ \ \ldots,\ \ (\sigma(2n-1),\sigma(2n),\ldots,\sigma(1),\sigma(2))\,\}$

$\mathcal{S}_2^n = \{\,(\sigma(1),\ldots,\sigma(2n)),\ \ \ldots,\ \ (\sigma(2),\sigma(1),\ldots,\sigma(2n),\sigma(2n-1))\,\}$

where $|\{[\sigma]\}| = |\mathrm{Aut}(\mathscr{D})|\,/\,|\mathrm{Aut}(D) = \mathcal{S}_n\times\mathcal{S}_2^n|$, by $\cong$ (*bijection*). $\square$

Lemma (orientation-parity ρ). *Let ρ = cycle orientation parity, such that X = quadratic lattice; if ρ = odd for all mesh, then ρ = odd (resp. even) for arbitrary cycle of even (resp. odd) vertices.*

Proof. $\heartsuit$.

Corollary. *Monomials of* $D_1, D_2 \subseteq \mathscr{D}$ *have equal sign* $\Longleftrightarrow$ *orientation parity is odd for all transition cycles of* $D_1 \Delta D_2 = D_1 \cup D_2 \backslash D_1 \cap D_2$, *where sign of monomial of* D *is given by*

$$\varepsilon_D^K = (-1)^{t(\sigma)} \prod_{\ell=1}^{n} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \quad \left| \begin{array}{l} \ell \subseteq D \subseteq \mathscr{D}, \ \forall \sigma \in \mathrm{Aut}(D) \\ t(\sigma) := (\sigma(1), \ldots, \sigma(2n)) \longrightarrow (1, \ldots, 2n). \end{array} \right.$$

Proof. By equivariance $(\varepsilon^K_{D_2})^{-1} \varepsilon^K_{D_1} : \mathrm{Aut}(D_1) \longrightarrow \mathrm{Aut}(D_2)$, if D_1, D_2 match $\sigma(2\ell-1)$ to $\sigma(2\ell)$, resp., $\tau(2\ell'-1)$ to $\tau(2\ell') \mid \ell, \ell' = 1, \ldots, n$, then

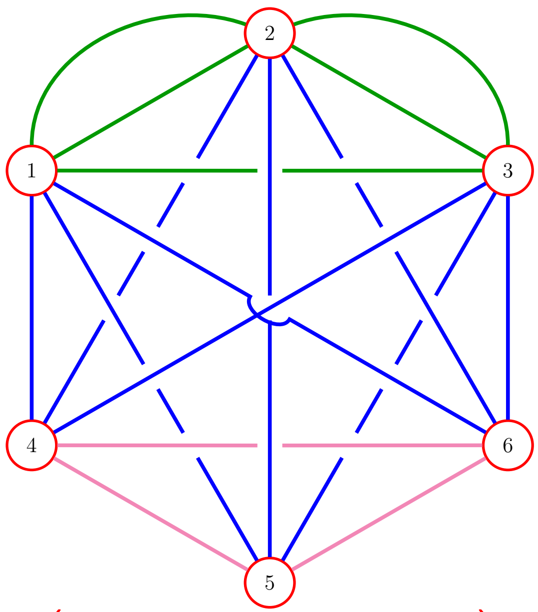
$$\varepsilon^K_{D_1} \varepsilon^K_{D_2} = (-1)^{t(\widetilde{\gamma})} \prod_{\ell=1}^{n} \varepsilon^K_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \prod_{\ell'=1}^{n} \varepsilon^K_{\widetilde{\tau}(2\ell'-1)\widetilde{\tau}(2\ell')} \quad \Big| \, \widetilde{\gamma} = \widetilde{\sigma} \circ \widetilde{\tau}$$

$$= \prod_{\ell=1}^{n} \varepsilon^K_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \prod_{\ell'=1}^{n} \varepsilon^K_{\widetilde{\tau}(2\ell'-1)\widetilde{\tau}(2\ell')} = (-1)^{\left(\sum_{C_\alpha} \widetilde{\gamma}_{D_1 \Delta D_2}(C_\alpha)\right)} \quad \left| \begin{array}{l} \text{by } \rho \\ \text{lemma} \end{array} \right.$$

$\Longleftrightarrow$ $\eta = \sum_{C_\alpha} \widetilde{\gamma}_{D_1 \Delta D_2}(C_\alpha)$, $\forall \, n^K_{C_\alpha}\big|_{\mathscr{D}} \pmod{\eta} =$ odd, given by

$$\Big(\sum_{C_\alpha} \sum_{\ell \subseteq \mathscr{D}} \mathbb{1}_{\{\varepsilon(\partial \mathcal{F}; i_\ell, j_\ell) \,=\, \varepsilon^K(\ell; j_\ell, i_\ell)\}} (C_\alpha) \Big) \pmod{\eta}$$

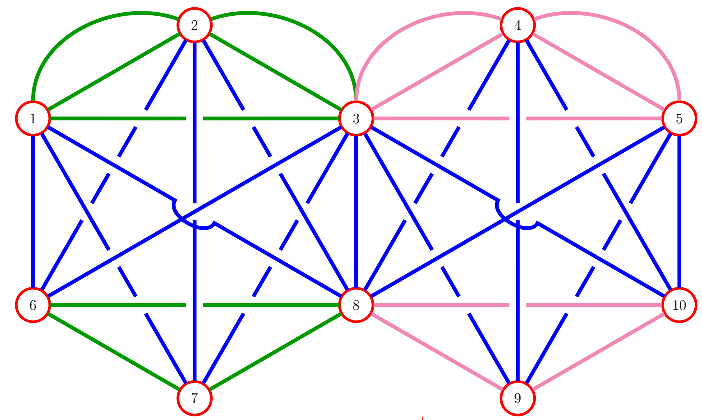
$\Longleftrightarrow$ $n^K_{C_\alpha}\big|_{C_\alpha} =$ odd, for all transition cycles $C_\alpha \mid \alpha = 1, \ldots, \eta$. $\square$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

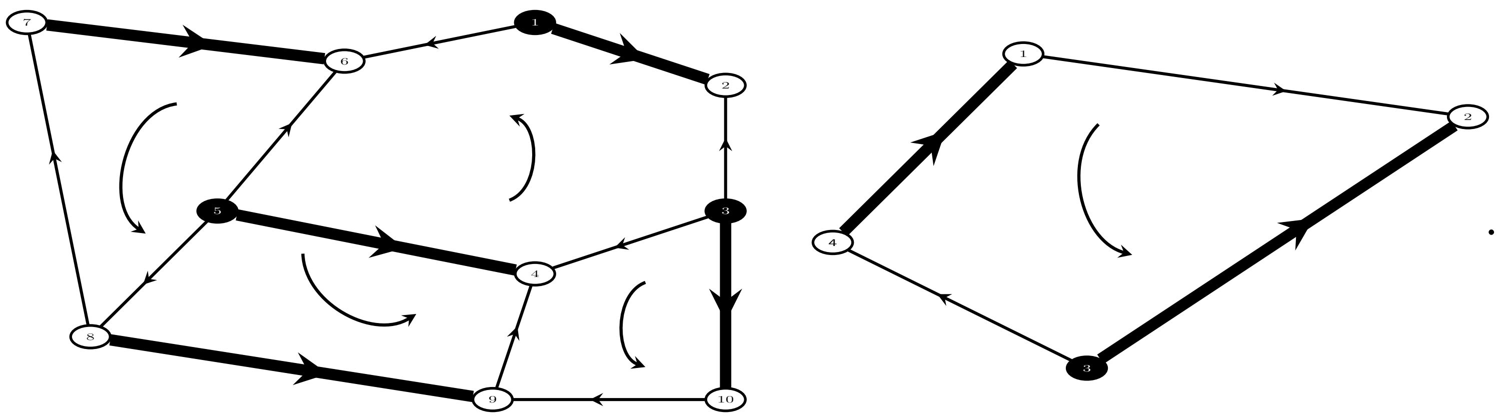
$\mathbf{0}$ = non-adjoined (i, j)

$\mathbf{1}, \mathbf{1}, \mathbf{1}$ = adjoined (i, j).



$$\left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

1.4 Kasteleyn orientation and matrix



Definition. Let $X \subset \overline{\mathcal{M}}_g = 1$-skeleton CW-complex (resp. genus g compact orientable surface cell-decomposition) embedding. Set of arbitrary orientation $\varepsilon^K(\ell; \cdot, \cdot)$, $\forall\, \ell \subset 2^X$, is Kasteleyn if with respect to a fixed (counterclockwise) i_ℓ to j_ℓ boundary orientation $\varepsilon(\partial\mathcal{F}; i_\ell, j_\ell)$, $\forall\, i_\ell \neq j_\ell$ faces $\mathcal{F}$, $\forall\, \ell \subseteq X$,

$$\prod_{\ell \in \partial\mathcal{F}} \varepsilon^K_{i_\ell j_\ell} = -1, \ \ \forall\, \mathcal{F} \subset 2^X \quad \Bigg| \quad \varepsilon^K_{i_\ell j_\ell} = \begin{cases} -1 & \text{if} \ \ \varepsilon(\partial\mathcal{F}; i_\ell, j_\ell) \ = \ \varepsilon^K(\ell; j_\ell, i_\ell) \\ +1 & \text{if} \ \ \varepsilon(\partial\mathcal{F}; i_\ell, j_\ell) \ = \ \varepsilon^K(\ell; i_\ell, j_\ell). \end{cases}$$

If X = Kasteleyn, weighted $\omega_{i_\ell j_\ell} = \omega_{j_\ell i_\ell} > 0$, $\forall\, \{i_\ell \neq j_\ell\}$ edges ℓ, then

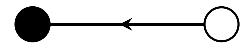
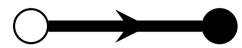
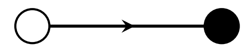
$$X^K_{ij} = \sum_\ell \varepsilon^K_{i_\ell j_\ell} \omega_{i_\ell j_\ell} = -X^K_{ji} \quad \Bigg| \quad X^K_{ij} = 0, \ \ \forall\, i_\ell, j_{\ell'} \,|\, \ell \neq \ell', \ \ \text{or} \ \ \forall\, i = j.$$

Remark.

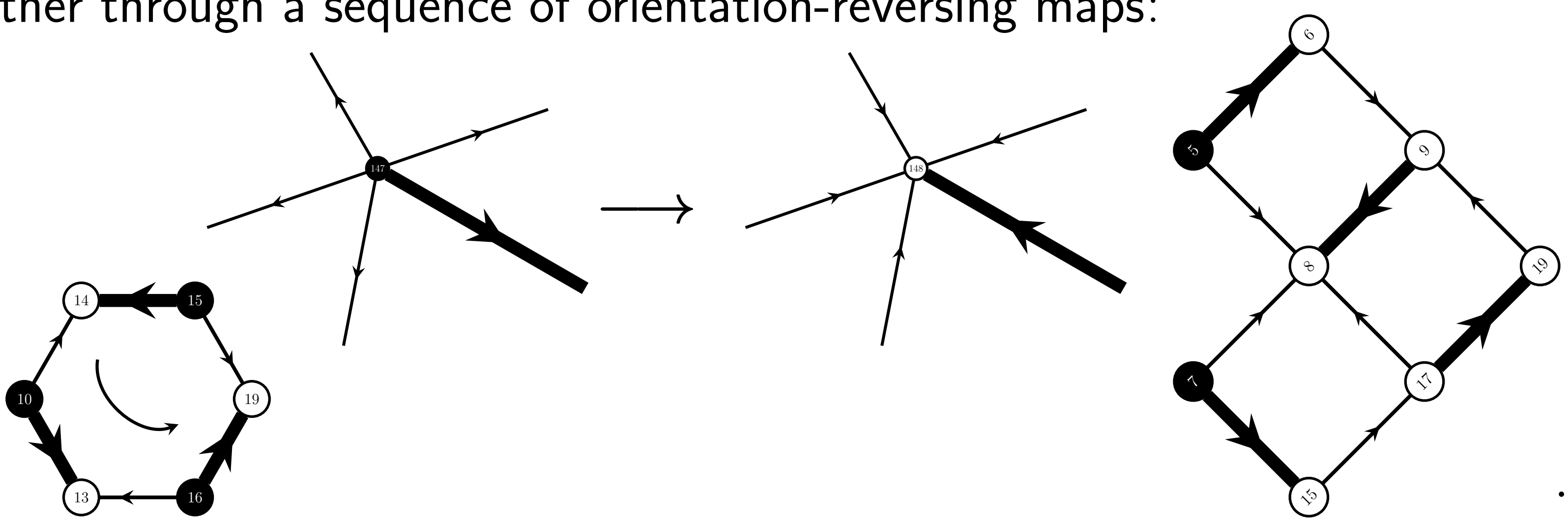
$$(X^K_{ij}) = \begin{cases} \text{Adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} = 1,\ \forall\, \varepsilon^K_{i_\ell j_\ell} = 1 \\ \text{Weighted adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} \neq 1,\ \forall\, \varepsilon^K_{i_\ell j_\ell} = 1 \\ \text{Skew-symmetric adjacency matrix} & \text{if } \omega_{i_\ell j_\ell} = 1,\ \forall\, \varepsilon^K_{i_\ell j_\ell} = -\varepsilon^K_{j_\ell i_\ell}. \end{cases}$$

Derivation (Kasteleyn). For counterclockwise $\varepsilon(\partial\mathcal{F})$ bipartite $2n$ lattice:

$X^K_{ij} = -X^K_{ji} =$

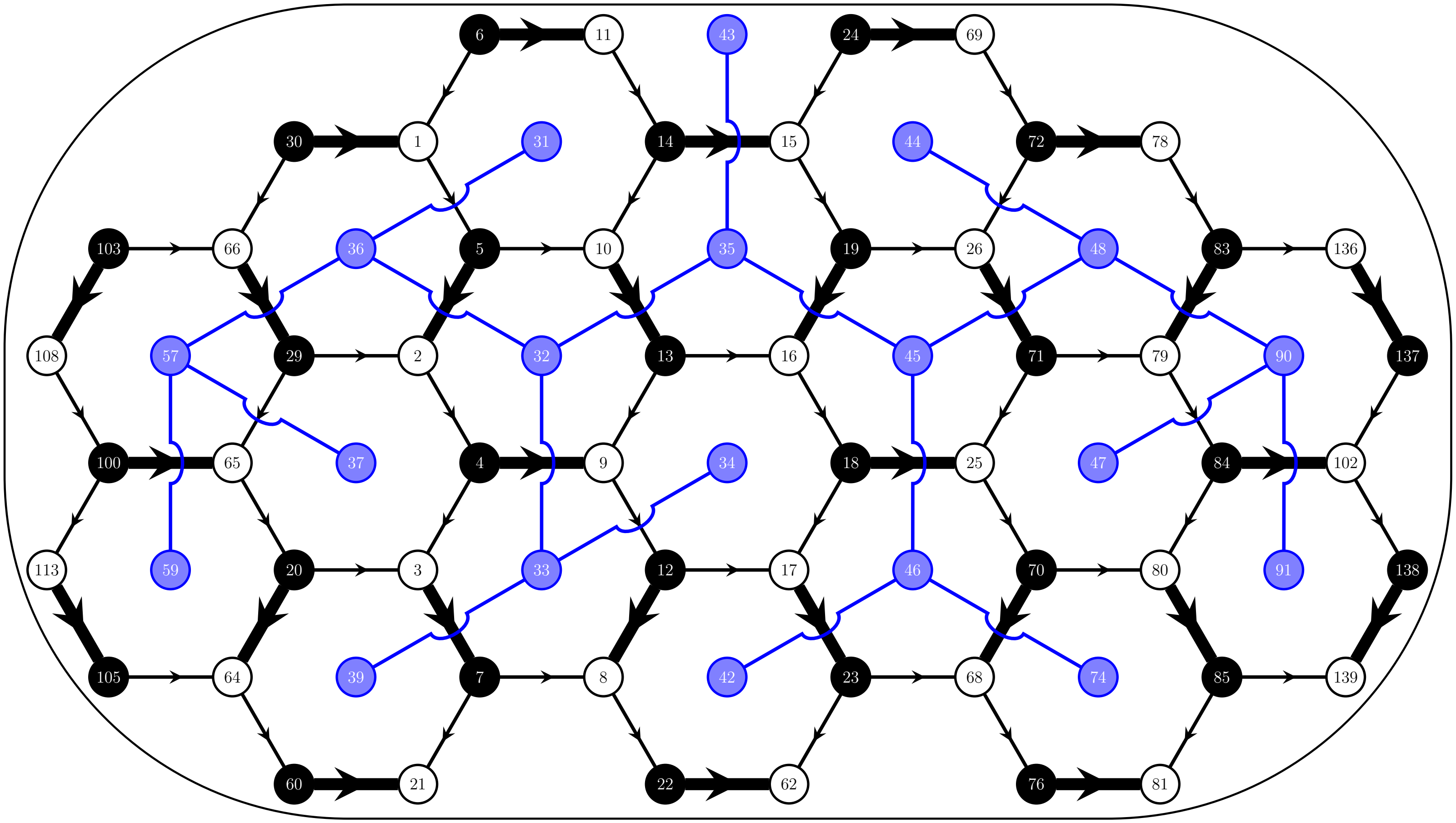
- $-\omega_{ij} = -\omega_{ji}$ if i_ℓ  j_ℓ or i_ℓ  j_ℓ
- $\omega_{ij} = \omega_{ji}$ if i_ℓ  j_ℓ or i_ℓ  j_ℓ
- 0 if $i = j$ or $\ell \neq \ell'$, $\forall\, i_\ell, j_{\ell'}$.

Definition. Two orientations are equivalent, if each is obtainable from the other through a sequence of orientation-reversing maps:



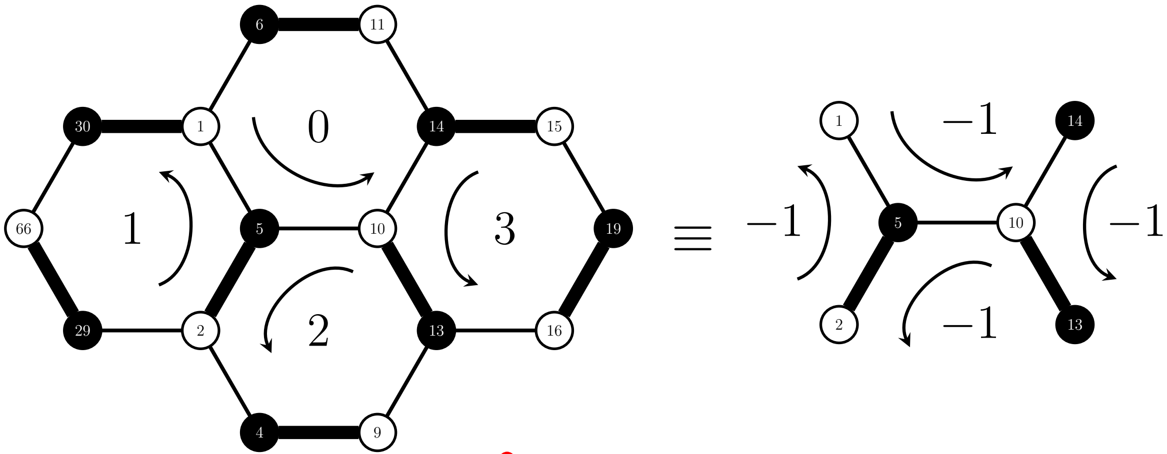
Lemma (existence). *Kasteleyn orientation exists.*

Proof. Following dual X^* spanning tree:

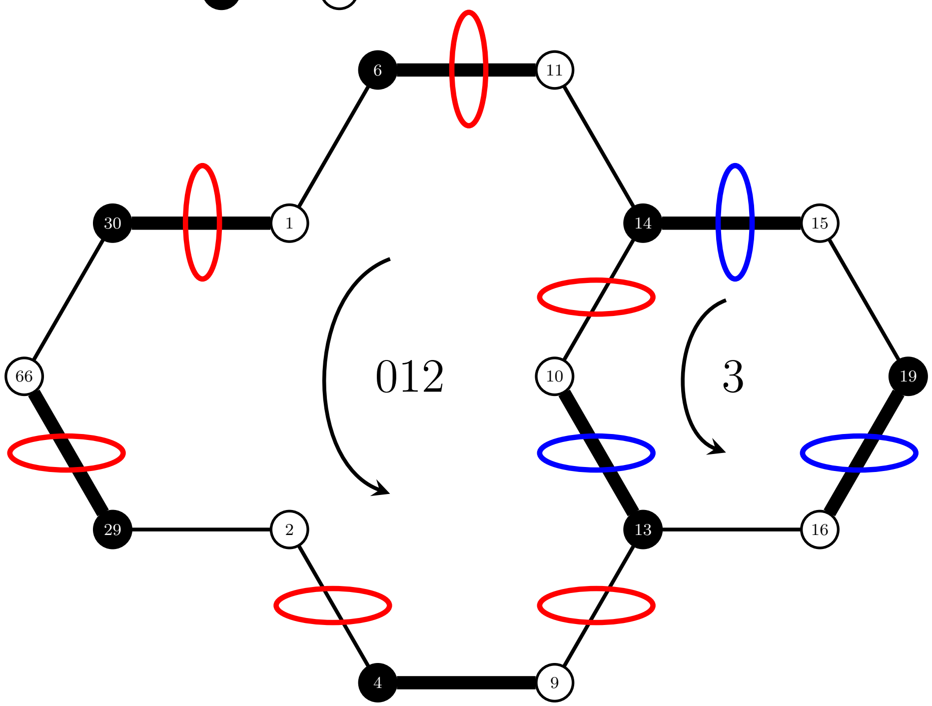


Reduce X to $\ll$; $n\times n \longrightarrow \exp(\alpha n^2)$. Arbitrarily orient all $\ell \subset 2^X$ not crossing X^* spanning tree rooted outside X. Deleting ℓ^* from leaves, make $\varepsilon^K(\mathcal{F})$.

Remark. Deleted-vertex changes Kasteleyn to non-Kasteleyn at "hole":



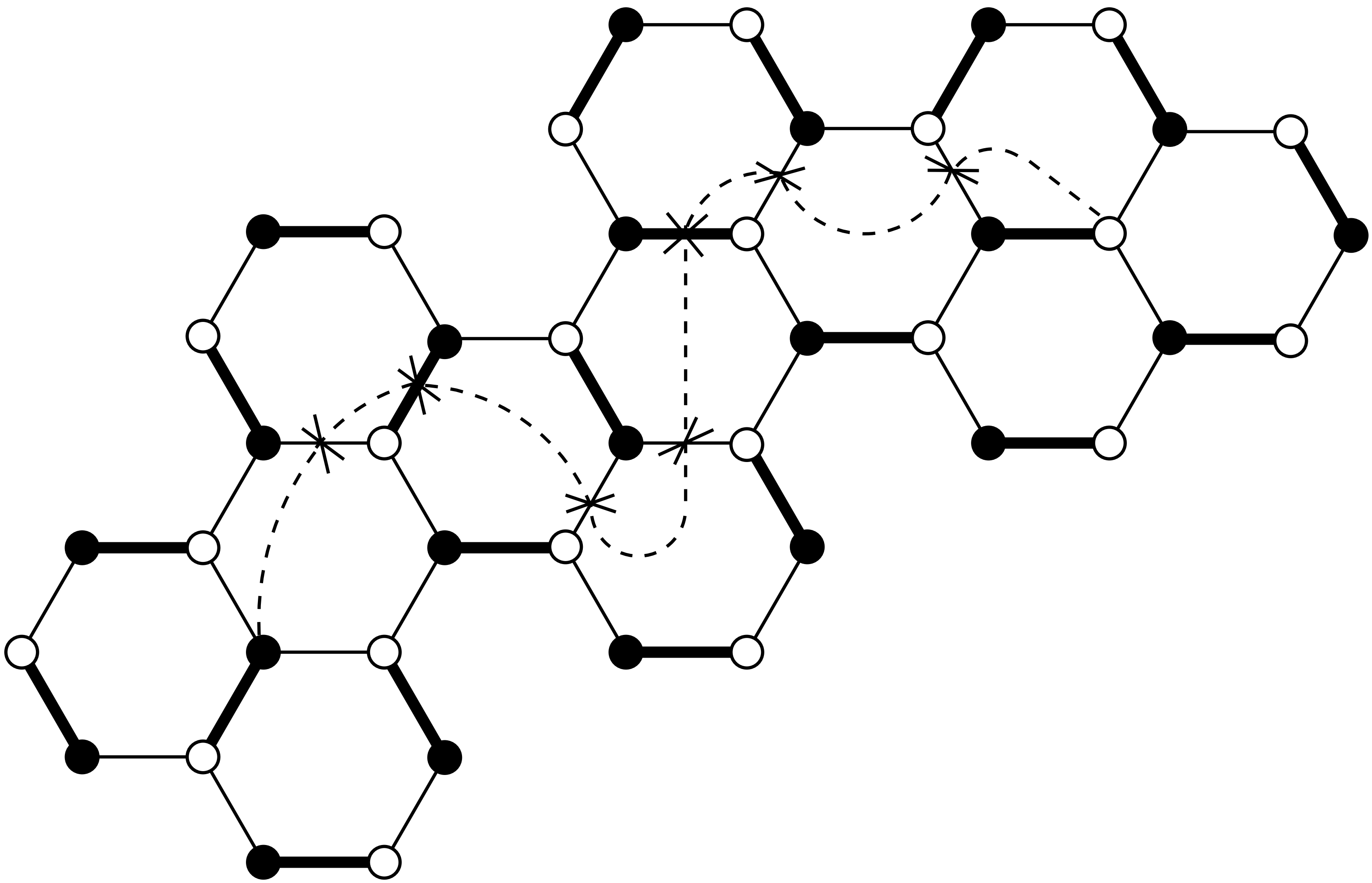
$h_0 = h_1 =$
$= h_2 = h_3 =$
$=$ Kasteleyn.



$h_{012} =$ non-Kasteleyn.

$h_3 =$ Kasteleyn.

Remark. To convert the non-Kasteleyn orientation back to Kasteleyn:



$$h_0 = h_1 = \cdots = h_{11} = -1.$$

Lemma. *All Kasteleyn orientations are equivalent for X planar.*

Proof. Consider arbitrary Kasteleyn orientations K and K': marked by K on ith end ℓ_-, K' on jth end ℓ_+, $\forall\, \ell \subset 2^X$, with respect to $\varepsilon(\partial\mathcal{F}; i_\ell, j_\ell)$.

The product

$$\prod_{\ell\,\in\,\partial\mathcal{F}} \varepsilon^{KK'}_{i_\ell j_\ell} = \prod_{\ell_-\,\in\,\partial\mathcal{F}} \varepsilon^{K}_{i_{\ell_-} j_{\ell_-}} \cdot \prod_{\ell_+\,\in\,\partial\mathcal{F}} \varepsilon^{K'}_{i_{\ell_+} j_{\ell_+}} = \prod_{\ell_-,\,\ell_+\,\in\,\partial\mathcal{F}} \varepsilon^{K}_{i_{\ell_-} j_{\ell_-}} \cdot \varepsilon^{K'}_{i_{\ell_+} j_{\ell_+}}$$

which equals

$$\prod_{\ell_-\,\in\,\partial\mathcal{F}} \varepsilon^{K}_{i_{\ell_-} j_{\ell_-}} \cdot \prod_{\ell_+,\,\ell_-\,\in\,\partial\mathcal{F}} \sigma^{KK'}_{\ell_+} \cdot \varepsilon^{K}_{i_{\ell_-} j_{\ell_-}} = (-1)\cdot(-1) = 1 \quad \Big|\, \sigma^{KK'}_{\ell_+} = \pm 1$$

where

$$\varepsilon^{KK'}_{i_\ell j_\ell} = \begin{cases} -1 & \text{if } \ \varepsilon^{K}(\ell_-; i_{\ell_-}, j_{\ell_-}) = \varepsilon^{K'}(\ell_+; j_{\ell_+}, i_{\ell_+}) \ \text{ i.e. } \ \sigma^{KK'}_{\ell_+} = -1 \\ +1 & \text{if } \ \varepsilon^{K}(\ell_-; i_{\ell_-}, j_{\ell_-}) = \varepsilon^{K'}(\ell_+; i_{\ell_+}, j_{\ell_+}) \ \text{ i.e. } \ \sigma^{KK'}_{\ell_+} = +1 \end{cases}$$

is well-defined for all K from K' and vice versa $K \longleftrightarrow K'$ in simple reversal of orientations around vertices, as required, by $\sigma^{KK'}_{\ell_+} = -1$. □

Lemma. *Equivalence class $[K]$ is unique for X planar.*

Proof. Follows from fundamental group being trivial for the plane. □

Theorem. Equivalence classes $\{[K]\}$ of Kasteleyn orientations equal 2^{2g}.

Proof. The equivalence classes $\{[K]\} \cong$ affine closure of non-degenerate skew-symmetric, quadratic form $q \in \mathsf{Sym}_k^2(V^\wedge)$ on characteristic-2 field k:

$$q(\alpha+\beta) = q(\alpha) + q(\beta) + \alpha\cdot\beta \;\Big|\, q: V\otimes V \longrightarrow k\,,\; \forall\, \alpha, \beta \in \mathcal{H}^1 = V\otimes V$$

$\forall\, \alpha \in \mathcal{H}^1 =$ first homology space, classified by:

$$\frac{1}{\sqrt{|\mathcal{H}^1|}} \sum_{q\,\in\,(\mathcal{H}^1,\cdot)} (-1)^{\mathit{Arf}(q)+q(\alpha)} = 1 \;\Big|\, \mathit{Arf}(q) = \sum_{\{e_i,\,e_j\}} q(e_i)q(e_j) \in (k/f(k)) \subset \mathbb{Z}_2$$

where $\{e_i, e_j\} =$ symplectic basis-pairs in symplectomorphisms $V \longrightarrow V$, for Lang's isogeny $f: k \longrightarrow k \;|\, x \longmapsto x^2 - x \in \mathbb{G}_a/\mathbb{F}_2$.

But, continuous $\psi: X \longrightarrow \overline{\mathcal{M}}_g \;|\, X \supseteq \psi$-faces $\mathcal{F} \approx$ open disk $=$ connected components of $\overline{\mathcal{M}}_g \backslash \psi(X) \Longrightarrow \exists\; \chi(X) = \chi(\overline{\mathcal{M}}_g)$ in Euler-Poincaré bound $|V| - |E| + |\mathcal{F}| = \chi(X) \geqslant \chi(\overline{\mathcal{M}}_g)$. And, all vanishing composition $\partial_1 \circ \partial_2$ of boundary operators $\partial_2 : C_2 \longrightarrow C_1$, $\partial_1 : C_1 \longrightarrow C_0, \forall\, C_0, C_1, C_2 =$ basis of 2D cell-complex vertices, edges, faces, resp., $\Longrightarrow$ 1-cycle space $\mathsf{Ker}(\partial_1)$ contains 1-boundary space $\partial_2(C_2)$. Hence, independent of X, and depending only on the genus g, $|\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)| = |\mathsf{Ker}(\partial_1)/\partial_2(C_2)| = 2^{2g}$. $\square$

Theorem (Kasteleyn). *Let* $X=(i_\ell,\forall\, i_\ell\neq j_{\ell'}\,|\,\ell\geqslant\ell';\, i\neq j)\subset\overline{\mathcal{M}}_g\,|\,g\gg$ *be embedding for all equivalence classes* $[\sigma]$ *of perfect matchings* $\mathscr{D}\supseteq D$, $\forall\,\sigma\in$ *Aut*$(\mathscr{D})$, *sgn*$(\sigma)=(-1)^{t(\sigma)}\,|\,t(\sigma):=(\sigma(1),\ldots,\sigma(2n))\longrightarrow(1,\ldots,2n)$; *the partition function* $Z=\sum_{D\subset\mathscr{D}}\prod_{\ell\subset D}\varepsilon_\ell^K\omega_\ell=\mathrm{Pf}(X^K)\in\mathbf{Quot}(\mathbb{K}[D])$ *is given by*

(i) $$\mathrm{Pf}(X^K)\;=\;\frac{1}{n!}\frac{1}{2^n}\sum_{\sigma\in\mathcal{S}_{2n}}(-1)^{t(\sigma)}\prod_{\ell=1}^{n}X^K_{\sigma(2\ell-1)\sigma(2\ell)}$$

(ii) $$\mathrm{Pf}(X^K)\;=\;\sum_{\sigma\big|_{[\sigma]}}(-1)^{t(\sigma)}\prod_{\ell=1}^{n}X^K_{\sigma(2\ell-1)\sigma(2\ell)}\,.$$

Proof. (i)$\Longrightarrow$(ii) by perfect-matching *bijection*. To see (i): $X^K = m\times m \Longrightarrow \det(X^K)^T = \det(-X^K) = (-1)^m \det X^K = 0 \Longleftrightarrow m =$ odd; $\det X^K \neq 0 \Longrightarrow \det X^K =$ positive-definite, square of rational function of $X^K_{ij} \mid X^K = 2n\times 2n$.

In particular, $X^K_{i\,\rho(i)} = -X^K_{\rho(i)\,i} \mid i \leqslant \rho(i) \Longrightarrow$ sum of 2-partition monomials:

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{\rho \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n\times\mathcal{S}_2^n)}} (-1)^{t(\rho)} \prod_{i=1}^{2n} X^K_{i\,\rho(i)} \quad \left| \begin{array}{l} j=\rho^{-1}(i) \longleftrightarrow i\neq j\in\{1,\ldots,n\} \\ \quad\Longrightarrow X^K_{i\,\rho(i)} \equiv X^K_{\rho(2\ell-1)\rho(2\ell)} \\ \quad \forall\,\ell=1,\ldots,2n; \\ t(\rho) = \text{even (odd), for even } 2n \text{ (otherwise)} \\ t(\rho) := (\rho(1),\ldots,\rho(2n)) \longrightarrow (1,\ldots,2n) \end{array}\right. \\ \qquad\qquad + \\ \displaystyle\sum_{\substack{\rho \\ \Cap \\ 2\cdot\left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}}\Big/\left(\mathcal{S}_2\times\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}-2\right\}}\right)\right)}} (-1)^{t(\rho)} \prod_{i=1}^{2n} X^K_{i\,\rho(i)} \quad \left| \begin{array}{l} j=\rho^{-1}(i) \longleftrightarrow i\neq j\in\{1,\ldots,n\} \\ \quad\Longrightarrow X^K_{i\,\rho(i)} \equiv X^K_{\rho(2\ell-1)\rho(2\ell)} \\ \quad \forall\,\ell=1,\ldots,2n; \\ t(\rho) = \text{odd (even)}, \\ \text{for even } 2n \text{ (otherwise).} \end{array}\right. \end{array}\right.$$

by Leibniz's second-index permutations.

Evidently, $X^K_{i\,\rho(i)} = -X^K_{\rho(i)\,i} \mid i \leqslant \rho(i) \Longrightarrow$ the quadratic multinomial:

$$\begin{cases} \displaystyle\sum_{\substack{\sigma = \widetilde{\sigma} \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n \times \mathcal{S}_2^n)}} (-1)^{t(\rho)+n+t(\sigma)} \left(\prod_{\ell=1}^{n} X^K_{\sigma(2\ell-1)\sigma(2\ell)} \right)^2 \quad \Bigg| \begin{array}{l} t(\rho) = \text{even (odd)}, \\ \text{for even } 2n \text{ (otherwise)} \end{array} \\ + \\ 2 \times \displaystyle\sum_{\substack{\begin{pmatrix} \sigma = \widetilde{\sigma} \neq \tau = \widetilde{\tau} \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n \times \mathcal{S}_2^n) \end{pmatrix} \\ \cong \\ \left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}} \Big/ \left(\mathcal{S}_2 \times \mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}-2\right\}} \right) \right)}} (-1)^{t(\sigma)+t(\tau)} \prod_{\ell=1}^{n} X^K_{\sigma(2\ell-1)\sigma(2\ell)} \prod_{\ell=1}^{n} X^K_{\tau(2\ell-1)\tau(2\ell)} \end{cases}$$

$$= \left(\sum_{\sigma = \widetilde{\sigma}} (-1)^{t(\sigma)} \prod_{\ell=1}^{n} X^K_{\sigma(2\ell-1)\sigma(2\ell)} \right)^2 = \mathsf{Pf}^2(X^K) \;\Bigg|\; \begin{array}{r} t(\sigma) := (\sigma(1), \ldots, \sigma(2n)) \\ \longrightarrow \; (1, \ldots, 2n). \end{array}$$

$$\forall \ \min(\deg(X)) \quad \Big| \ \{[\sigma]\} \subseteq \mathcal{S}_{2n}/(\mathcal{S}_n \times \mathcal{S}_2^n).$$

Now, write Z by all non-vanishing monomials $\forall\, D$:

$$Z \;=\; \sum_{\sigma=\widetilde{\sigma}} \sum_{D} \prod_{\ell=1}^{n} \varepsilon^{K}_{\sigma^D(2\ell-1)\sigma^D(2\ell)}\; \omega_{\sigma^D(2\ell-1)\sigma^D(2\ell)}$$

where, if D and $\widetilde{\sigma}$ are 1-1, then $Z = \sum_{\sigma=\widetilde{\sigma}} \prod_{\ell=1}^{n} \varepsilon^{K}_{\sigma(2\ell-1)\sigma(2\ell)} \omega_{\sigma(2\ell-1)\sigma(2\ell)}$.

But, for any $\{\sigma\,\big|_{[\sigma]}\}$ = set of all partitions of disjoint equivalence classes:

$$\mathrm{Pf}(X^K) = \sum_{\sigma\big|_{[\sigma]}} \underbrace{(-1)^{t(\widetilde{\sigma})-\Delta(\sigma)} \prod_{\ell=1}^{n} X^{K}_{\sigma(2\ell-1)\sigma(2\ell)}}_{D\text{-fixed } \pm,\ \forall\,\sigma\big|_{[\sigma]}} \quad \left|\; \begin{array}{l} \Delta(\sigma) := \varepsilon^{K}_{\sigma(2\ell-1)\sigma(2\ell)} \\ \qquad\qquad \longrightarrow \varepsilon^{K}_{\sigma(2\ell)\sigma(2\ell-1)} \\ \forall\, \sigma(2\ell-1) > \sigma(2\ell) \end{array}\right.$$

$$= \left\{ \frac{1}{n!}\,\frac{1}{2^n} \sum_{\substack{\sigma \\ \Cap \\ \mathrm{Aut}(\mathscr{D})}} \sum_{D} \overbrace{(-1)^{t(\sigma^D)} \prod_{\ell=1}^{n} \varepsilon^{K}_{\sigma^D(2\ell-1)\sigma^D(2\ell)}}^{D\text{-fixed } \pm,\ \forall\,\sigma^D\big|_{[\sigma^D]}} \prod_{\ell=1}^{n} \omega_{\sigma^D(2\ell-1)\sigma^D(2\ell)}. \right.$$

That is, such that all $\mathcal{S}_{2n}\backslash\mathsf{Aut}(\mathscr{D})$ monomials vanish, write:

$$\sum_{D\subseteq\mathscr{D}}\prod_{\ell\in D}\varepsilon_\ell^K\omega_\ell = \mathsf{Pf}(X^K),$$

by (i) and (ii)

the *bijection* $(-1)^{t(\widetilde{\sigma}^D)} = (-1)^{t(\sigma^D)+\Delta(\sigma^D)}$ equals

$$\frac{1}{n!}\frac{1}{2^n}\sum_{\sigma^D\in(\mathcal{S}_n\times\mathcal{S}_2^n)\big|_{[\sigma^D]}}(-1)^{t(\sigma^D)}\prod_{\ell=1}^{n}\varepsilon^K_{\sigma^D(2\ell-1)\sigma(2\ell)}$$

i.e. depending only on perfect-matching orientation; independent of $\widetilde{\sigma}^D$. □

Corollary. *Correlation* $=$ *Pfaffian of inverse Kasteleyn operator*

$$\left\langle\prod_{i=1}^{k}\sigma_D(\ell_i)\right\rangle = \mathsf{Pf}\big((X^K)^{-1}_{\xi\eta}\big)\quad\Big|\ \xi,\eta=1,\ldots,k;\ \mathsf{Pf}(X^K)=\textit{partition function}.$$

Proof. Follows from Kasteleyn theorem for $n=k$. □

Remark. Combinatorial (exponential) complexity reduces to cubic complexity of diagonalizing $\mathsf{Pf}(\mathcal{A}X^K\mathcal{A}^T)=\det(\mathcal{A})\mathsf{Pf}(X^K)\longrightarrow\mathcal{O}(n^3)$ by skew symmetric Gaussian elimination. And, the discrete property, correlation, is universal $\forall\, g$ since behavior of local observable is determined at point.

1.5 Computing isomorphisms

For fixed genus g, number of isomorphisms, to wit, equivalence classes $[\sigma]$, are enumerated in Z as two-variable skew-symmetric adjacency $z_1 = 1 = z_2$ matching polynomial, by the generating function

$$\sum_{D(N_1,\ldots,N_k)\ \mid\ (\sum_{\nu=1}^{k} N_\nu)=n} (\pm) \prod_{\nu=1}^{k} z_\nu^{N_\nu} \quad \Big| N_\nu = |\text{edges}| \text{ of } z_\nu\text{-class.}$$

Derivation I. Let $X \subset \overline{\mathcal{M}}_g =$ planar $M \times N$ square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X;\, M, N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\cos^2\left(\frac{\pi i}{M+1}\right) + \cos^2\left(\frac{\pi j}{N+1}\right)} \quad \Big| N = \text{even}$$

$$= |\{\widetilde{\sigma}(X;\, N, M)\}| \quad \Big| \begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big| MN = \text{odd.}$$

Show. ♡.

Derivation II. Let $X \subset \overline{\mathcal{M}}_g =$ cylindrical $M \times N$ square grid.

$$|\{\widetilde{\sigma}(X;\, M, N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\big(\frac{\pi(2i-1)}{M}\big) + \cos^2\big(\frac{\pi j}{N+1}\big)} \quad \Big| N = \text{even}$$

$$= 2^{(\frac{MN}{2} - \frac{M}{2} + 1)} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\big(\frac{\pi(2i-1)}{M}\big) + \cos^2\big(\frac{\pi j}{N+1}\big)} \quad \Big| \begin{matrix} M = \text{even} \\ N = \text{odd} \end{matrix}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. $\heartsuit$.

Derivation III. Let $X \subset \overline{\mathcal{M}}_g =$ toroidal $M \times N$ square grid.

$$|\{\widetilde{\boldsymbol{\sigma}}(X;\, M,N)\}| =$$

$$= 2^{(\frac{MN}{2}-1)} \left(\begin{array}{c} \displaystyle\prod_{i=1}^{M}\prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \sin^2\left(\frac{2\pi j}{N}\right)} \\ + \\ \displaystyle\prod_{i=1}^{M}\prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{2\pi i}{M}\right) + \sin^2\left(\frac{\pi(2j-1)}{N}\right)} \\ + \\ \displaystyle\prod_{i=1}^{M}\prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{M}\right) + \sin^2\left(\frac{\pi(2j-1)}{N}\right)} \end{array} \right) \Bigg|\, N = \text{even}$$

$$= |\widetilde{\boldsymbol{\sigma}}(X;\, N,M)| \quad \Bigg|\begin{array}{l} M = \text{even} \\ N = \text{odd} \end{array}$$

$$= 0 \quad \Big|\, MN = \text{odd}.$$

Show. ♡.

Derivation IV. Let $X \subset \overline{\mathcal{M}}_g =$ planar 6×8 square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X;\, M, N)\}| =$$

$$= 16777216\, \Big(\frac{1}{4} + \cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{9}) + \cos^2(\frac{\pi}{7})\Big)\Big(\cos^2(\frac{\pi}{7}) + \cos^2(\frac{2\pi}{9})\Big) \times$$
$$\times \Big(\cos^2(\frac{\pi}{7}) + \sin^2(\frac{\pi}{18})\Big)\Big(\frac{1}{4} + \sin^2(\frac{\pi}{14})\Big)\Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{\pi}{14})\Big) \times$$
$$\times \Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{\pi}{14})\Big)\Big(\frac{1}{4} + \sin^2(\frac{3\pi}{14})\Big) \times$$
$$\times \Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big)\Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big)\Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{3\pi}{14})\Big).$$

Show. $\heartsuit$.

Derivation V. Let $X\subset\overline{\mathcal{M}}_g=$cylindrical 6×8 square grid.

$$|\{\widetilde{\sigma}(X;\, M,N)\}| =$$

$$= 5242880\,\Big(\frac{1}{4}+\cos^2(\frac{\pi}{9})\Big)^2\Big(1+\cos^2(\frac{\pi}{9})\Big)\Big(\frac{1}{4}+\cos^2(\frac{2\pi}{9})\Big)^2\times$$

$$\times\,\Big(1+\cos^2(\frac{2\pi}{9})\Big)\Big(\frac{1}{4}+\sin^2(\frac{\pi}{18})\Big)^2\Big(1+\sin^2(\frac{\pi}{18})\Big)$$

Show. ♡.

Derivation VI. Let $X\subset\overline{\mathcal{M}}_g=$toroidal 6×8 square grid.

$$|\{\widetilde{\sigma}(X;\, M,N)\}| =$$

$$= 8388608\,\Big[\,\frac{18225}{131072}\,+\,\cos^4(\frac{\pi}{8})\Big(\frac{3}{4}+\cos^2(\frac{\pi}{8})\Big)^4\sin^4(\frac{\pi}{8})\Big(\frac{3}{4}+\sin^2(\frac{\pi}{8})\Big)^4+$$

$$+\,\Big(\frac{1}{4}+\cos^2(\frac{\pi}{8})\Big)^4\Big(1+\cos^2(\frac{\pi}{8})\Big)^2\Big(\frac{1}{4}+\sin^2(\frac{\pi}{8})\Big)^4\Big(1+\sin^2(\frac{\pi}{8})\Big)^2\Big].$$

Show. ♡.

1.6 Grassmann integral

Definition. The Grassmann algebra $\bigwedge^{\bullet} V$ on a basis $(a_1, \ldots, a_{2n})$ of V is generated by $2^{2n} = \sum_{k=0}^{2n} (\dim \bigwedge^k V) = \sum_{k=0}^{2n} \binom{2n}{k}$ dimensional basis vectors

$$\left\{ \begin{array}{l} a_0 = 1;\ a_{\sigma(k)<} = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},\ \forall\, \sigma(k)< \,=\, (\sigma(1), ..., \sigma(k)) \mid \sigma(1) < \cdots < \sigma(k); \\ a_{\sigma(\xi)} \otimes a_{\sigma(\eta)} + a_{\sigma(\eta)} \otimes a_{\sigma(\xi)} = 0, \forall\, \xi, \eta = 1, \ldots, k;\ \sigma(\xi), \sigma(\eta), k = 1, \ldots, 2n \end{array} \right\}$$

with:

(i) Element, k-component $(\forall\, k = 0, 1, \ldots, 2n)$

$$\bigwedge^k V \colon \bigotimes^k V \longrightarrow \bigotimes^k V \left| \begin{array}{l} \left(v^{\sigma(k)<} = \dfrac{1}{k!} \displaystyle\sum_{\sigma(k)<} \sum_{\sigma} (-1)^{t(\sigma)} \prod_{i=1}^{k} v_{i\, \sigma(i)} \right) a_{\sigma(k)<} \\ \sigma \in \mathcal{S}_{\sigma(k)<}\ ,\ \ t(\sigma) := (\sigma(1), ..., \sigma(k)) \longrightarrow \sigma(k)< \end{array} \right.$$

(ii) Multiplication $(\forall\, k, l = 0, 1, \ldots, 2n)$

$$\begin{array}{l} (vw)^{\sigma(k)<\ \sigma(l)<} \;=\; v^{\sigma(k)<}\, w^{\sigma(l)<} \\ (vw)^0 \;=\; v_0 w_0 \end{array} \left| \begin{array}{l} \sigma(i)\big|_{\sigma(k)<} = \sigma(j)\big|_{\sigma(l)<} \Longrightarrow 0 \\ \forall\, \bigwedge^k V,\ \bigwedge^l V;\ k, l = 1, \ldots, 2n \end{array} \right.$$

$$\forall\, v \in \mathrm{Span}(\bigwedge^{\bullet} V) \colon \left\{ v = v_0 + \sum_{k=1}^{2n} v_k\, a_k + \sum_{k=2}^{2n} v^{\sigma(k)<}\, a_{\sigma(k)<} \ \middle|\ v_{(\cdot)} \in \mathbb{C} \right\}.$$

Definition. $\bigwedge^\bullet V$ integral := with respect to orientation $\mathbb{R} \simeq x \in \bigwedge^{2n} V$:

$$\int_{\bigwedge^{2n} V} f \;=\; f_x \qquad \Big|\, f = f_x\, x + \underbrace{\cdots}_{\text{lower order terms}}$$

and, if (a_i) is a basis in V, then $x = a_1 \otimes \cdots \otimes a_n$ by the formal constraints:

(i) $$\int \Big(\bigotimes_{i=1}^{k} a_{\sigma(i)}\Big) \otimes da = \begin{cases} 0 & \big| k < 2n \\ (-1)^{t(\sigma)} & \big| k = 2n \end{cases} \qquad \Bigg| \begin{aligned} da \;&\cong\; (-1)^{n(2n-1)} \bigotimes_{i=1}^{2n} da_i \\ t(\sigma) &:= (\sigma(1), \ldots, \sigma(2n)) \\ &\longrightarrow\; (1, \ldots, 2n). \end{aligned}$$

(ii) $$\int \Big(\bigotimes_{i=1}^{2n} a_i\Big) \otimes \Big(\bigotimes_{i=1}^{2n} da_i\Big) \;=\; (-1)^{n(2n-1)} \int \bigotimes_{i=1}^{2n} \Big(a_i \otimes da_i\Big) \;=\; (-1)^{n(2n-1)}.$$

Lemma. *$\bigwedge^\bullet V$ graded identity, up to tensors in superalgebra $M_{a,b}$ of minimal subfield, is isomorphic to kernel of either $\mathbb{Q}$ or prime-ordered field $\mathbb{F}_{q=p^m}$.*

Proof. $\heartsuit$.

Theorem. *T-ideal of $M_{pr+qs,\, ps+qr}$ is contained in T-ideal of $M_{p,q} \otimes M_{r,s}$.*

Proof. Follows from the prior lemma.

Theorem. *Let* $A^*(a) = \int_{\bigwedge^\bullet V} A(a) \ \big| A(a) = \exp\big(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\big),\ (a_i) \subseteq V,$ *satisfy the Grassmann constraints*; A^* *uniquely maximizes* $-\int_{\bigwedge^\bullet V} A \log A$ *over all* A *satisfying the integral such that*:

(i) $\mathsf{Pf}(A) = \displaystyle\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\Big) da$

(ii) $\mathsf{Pf}\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = \det(A)$

(iii) $(\mathsf{Pf}(A))^2 = \det(A)$

(iv) $\dfrac{\partial}{\partial A_{i_1 j_1}} \cdots \dfrac{\partial}{\partial A_{i_k j_k}} \mathsf{Pf}(A) =$

$$= \mathsf{Pf}(A) \cdot \mathsf{Pf}((A^{-1})_{ab}) \ \Big|\begin{matrix} a = i_1, \ldots, i_k \\ b = j_1, \ldots, j_k\,. \end{matrix}$$

Proof (hints).

(i). Write:

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)\, da \;=\; \frac{1}{n!}\,\frac{1}{2^n} \int_{\bigwedge^\bullet V} \langle a, Aa\rangle^n\, da$$

such that

$$\int \langle a, Aa\rangle^{2n}\, da \;=\; \int a_{\sigma(1)} a_{\tau(1)} \cdots a_{\sigma(n)} a_{\tau(n)}\, A_{\sigma(1)\,\tau(1)} \cdots A_{\sigma(n)\,\tau(n)}\, da \;=$$

$$= \;(-1)^{t(\sigma)} A_{\sigma(1)\,\tau(1)} \cdots A_{\sigma(n)\,\tau(n)} \qquad \Big| \; t(\sigma) : \; (\sigma(1), \tau(1), \ldots, \sigma(n), \tau(n)) \longrightarrow (1, \ldots, 2n).$$

This implies

$$\int_{\bigwedge^\bullet V} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big)\, da \;=\; \frac{1}{n!}\,\frac{1}{2^n}\, \mathsf{Pf}(A).$$

Remark. II, III and IV follow from the latter integral formula.

(ii). Choosing splitting $V = W \oplus W^*$ by matrix block structure, where V Grassmann algebra is isomorphic to algebra (tensor product) generated by $c_i, b_i \,|\, i=1,\ldots,n$ with relations $c_i c_j = -c_j c_i$, $c_i b_j = -b_j c_i$, and $b_i b_j = -b_j b_i$:

$$
\begin{aligned}
(a_1, \ldots, a_{2n}) &= \\
&= (\underbrace{c_1, \ldots, c_n}_{\text{basis in } W}, \underbrace{b_1, \ldots, b_n}_{\text{basis in } W^*}).
\end{aligned}
$$

As a result,

$$
\left\langle a\,,\ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} a \right\rangle \;=\; 2\,\langle c, Ab\rangle.
$$

Therefore, prove

$$
\int\limits_{\bigwedge^n(W \oplus W^*)} \exp\left(\langle c, Ab\rangle\right) dc\, db \;=\; \det(A).
$$

(iii). Similar.

(iv). $$\int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle a, \eta\rangle\Big) da \quad =$$

$$= \quad \int \exp\Big(\frac{1}{2}\big\langle a+A^{-1}\eta\,,\ A(a+A^{-1}\eta)\big\rangle \ - \ \frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big) da$$

$$= \quad \mathsf{Pf}(A)\ \exp\Big(-\frac{1}{2}\big\langle \eta, A^{-1}\eta\big\rangle\Big).$$

$$\frac{\partial}{\partial A_{i_1 j_1}} \cdots \frac{\partial}{\partial A_{i_k j_k}}\ \mathsf{Pf}(A) \quad =$$

$$= \quad \int a_{i_1} a_{j_1} \cdots a_{i_k} a_{j_k} \exp\Big(\frac{1}{2}\langle a, Aa\rangle\Big) da$$

$$= \quad \Big(\frac{\partial}{\partial \eta}\Big)^{2k} \int \exp\Big(\frac{1}{2}\langle a, Aa\rangle + \langle \eta, a\rangle\Big) da. \qquad \square$$

Theorem (bipartite). *Let* $X \subset \mathbb{R}^2 =$ *Kasteleyn-oriented bipartite partition*,

$$Z_X = \varepsilon_X^K \int \exp\Big(\frac{1}{2}\sum_{ij} a_i\,(X_{ij}^K)\,a_j\Big)da \;\left|\begin{array}{l} \varepsilon_X^K = (-1)^{\sigma}\,\varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{2n-1}\sigma_{2n}} \in \{\pm 1\} \\ 2n = |V(X)|. \end{array}\right.$$

Proof. $X =$ bipartite implies

$$X^K = \begin{pmatrix} 0 & B_X^K \\ -(B_X^K)^T & 0 \end{pmatrix} \quad \left|\begin{array}{l} B^K:\ \mathbb{R}^{V_\circ(X)} \longrightarrow \mathbb{R}^{V_\bullet(X)} \\ \mathbb{R}^{V(X)} = \mathbb{R}^{V_\bullet(X)} \oplus \mathbb{R}^{V_\circ(X)} \\ \dim(\mathbb{R}^{V_\bullet(X)}) = \dim(\mathbb{R}^{V_\circ(X)}) = n \\ V(X) = V_\bullet(X) \sqcup V_\circ(X),\ \ |V(X)| = 2n. \end{array}\right.$$

Identifying $V_\bullet(X)$, $V_\circ(X)$ via a diagram $\{b\} \sim \{\omega\}$ with "hole"

$$X^K = \begin{pmatrix} 0 & C_X^K \\ -(C_X^K)^T & 0 \end{pmatrix} \quad \left|\begin{array}{l} \mathbb{R}^{V(X)} = \mathbb{R}^{V_\bullet(X)} \oplus \mathbb{R}^{V_\circ(X)} \hookleftarrow \\ C_X^K = \mathbb{R}^{V_\circ(X)} \hookleftarrow \\ \hookleftarrow \implies \text{recursion i.e. nested.} \end{array}\right.$$

As a result, $Z_X = |\det(C_X^K)|$. $\square$

Corollary (bipartite correlation).

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle = \frac{\partial}{\partial\, \omega(b_1 w_1)} \cdots \frac{\partial}{\partial\, \omega(b_k w_k)} \ln Z_X$$

$$\Longrightarrow \langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle = \det\Big(\big((C_X^K)^{-1} \big)_{\widetilde{b}\, w} \Big) \quad \left| \begin{array}{l} \widetilde{b} = \widetilde{b}_1, \ldots, \widetilde{b}_k \\ w = w_1, \ldots, w_k \end{array} \right.$$

where $\widetilde{b}$ = white-vertex identified with b.

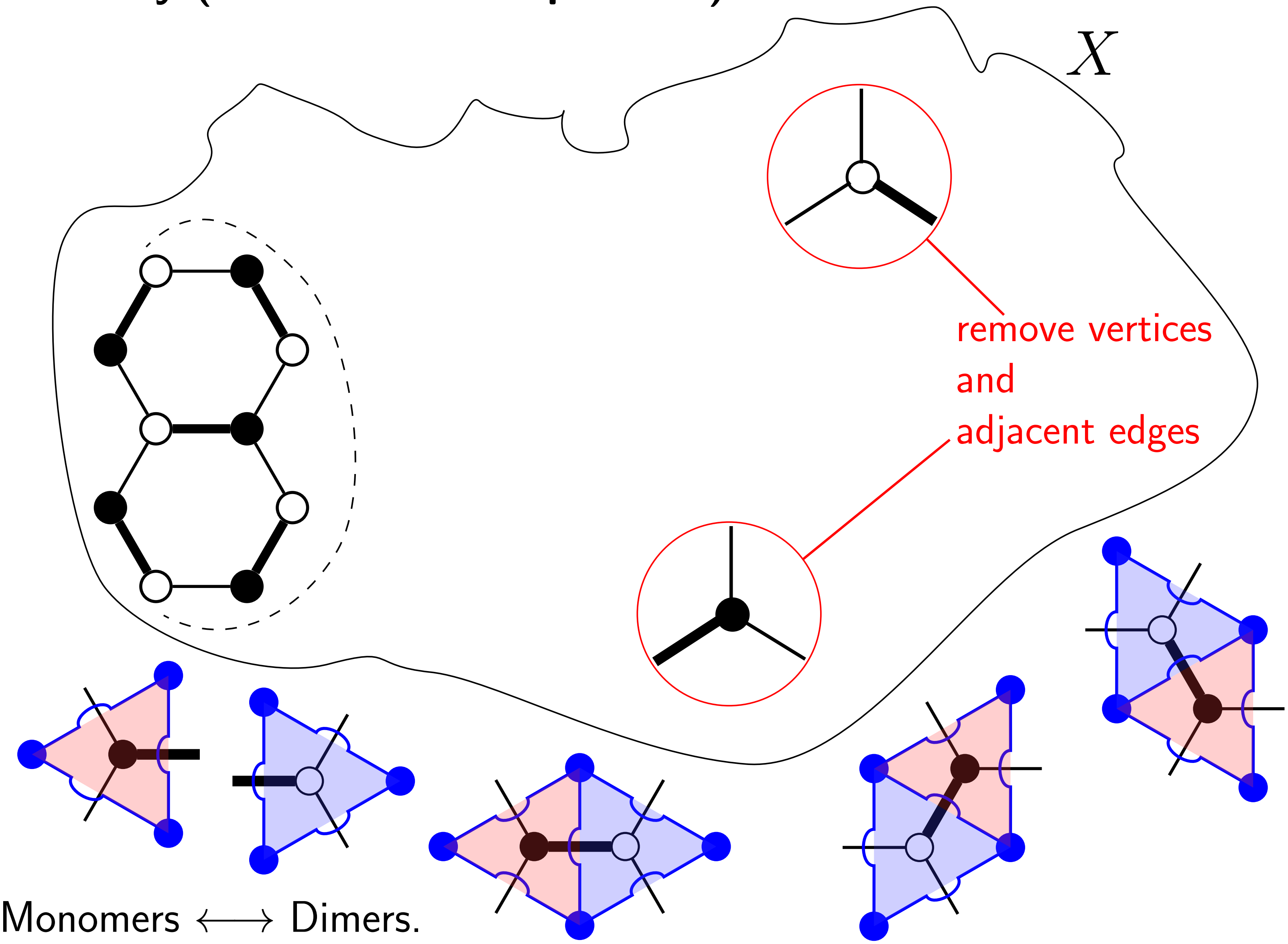
Remark. The "physical" meaning of (ii):

$$\langle \sigma_{b_1 w_1} \cdots \sigma_{b_k w_k} \rangle =$$

$$= \int \psi^*_{b_1} \psi_{w_1} \cdots \psi^*_{b_k} \psi_{w_k} \exp\big(\psi^* C_X^K \psi\big) d\psi^* d\psi \;\cdot \Big(\int \exp\big(\psi^* C_X^K \psi\big) d\psi^* d\psi \Big)$$

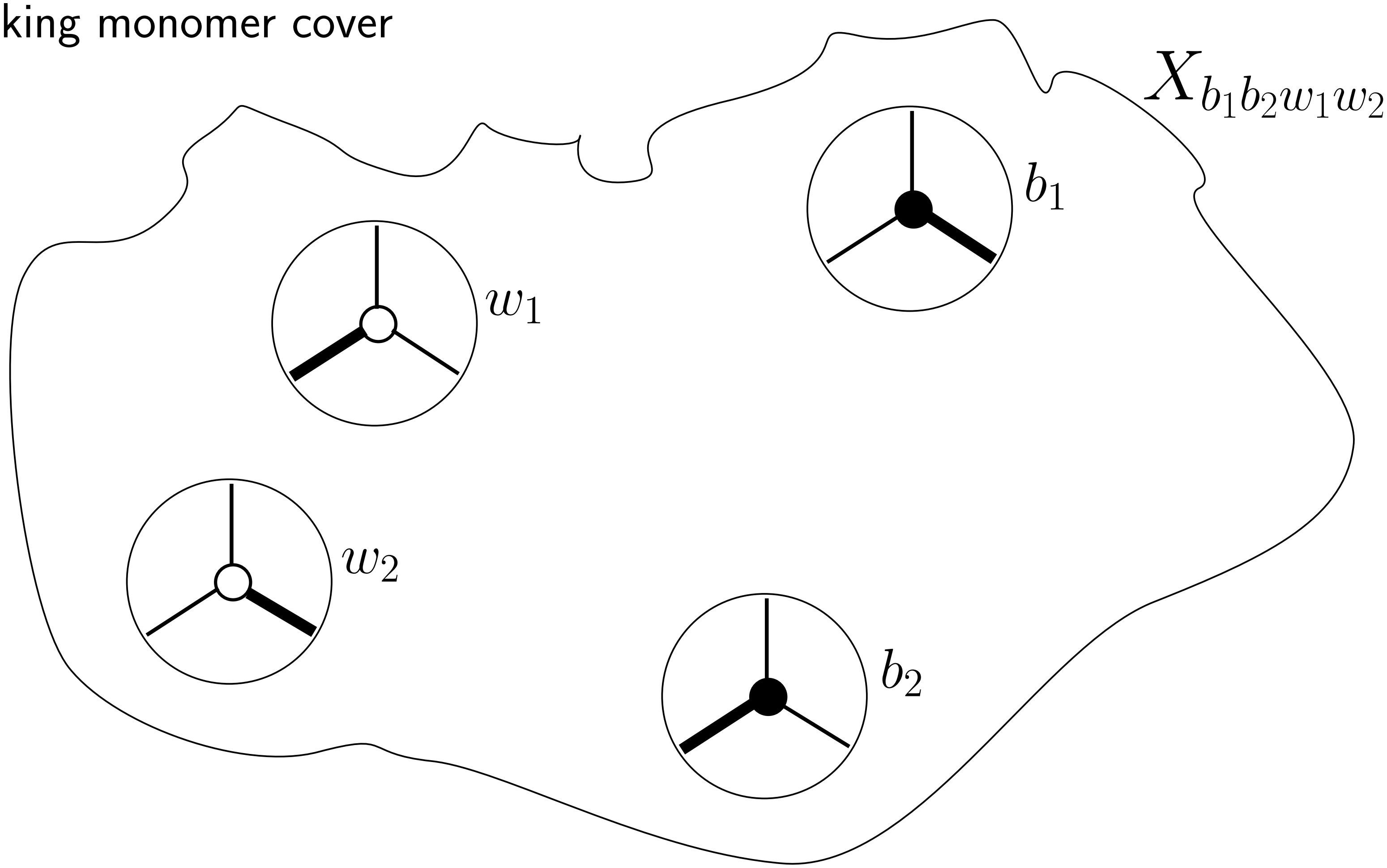
which corresponds to correlation for the free Fermionic representation.

Corollary (dimer-monomer problem).



Monomers $\longleftrightarrow$ Dimers.

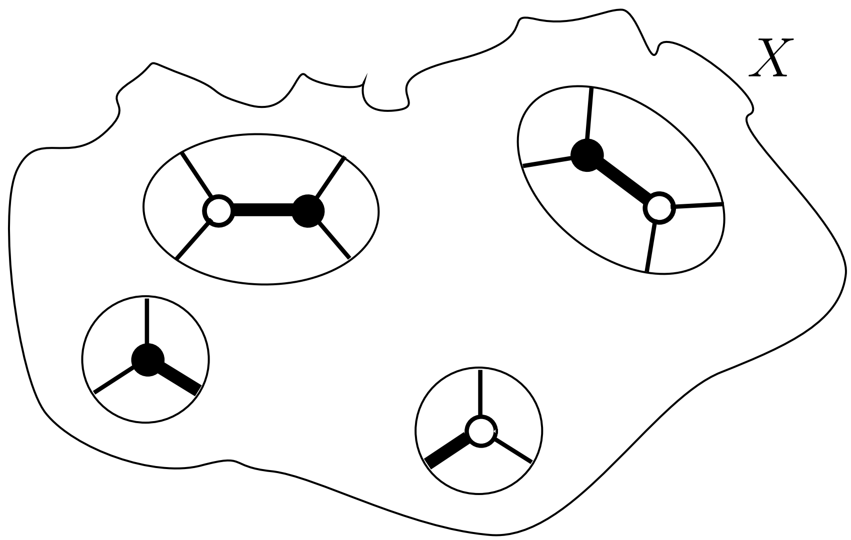
Taking monomer cover



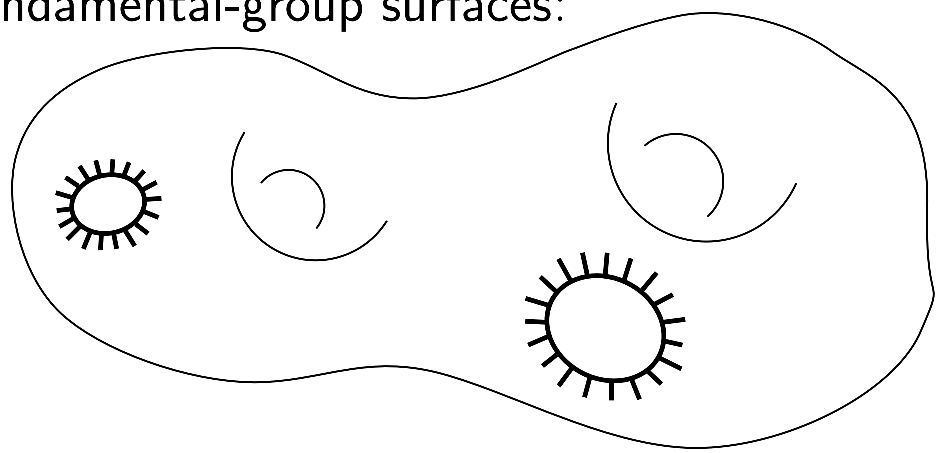
the monomer-monomer correlation $M_{b_1 \cdots b_n w_1 \cdots w_n}$ is given by

$$\frac{Z_{X_{b_1 \cdots b_n w_1 \cdots w_n}}}{Z_X}.$$

In particular, adjacent monomers $(b_\ell, w_\ell) \Longrightarrow$ dimer $(i_{b_\ell}\, j_{w_\ell})$, $\forall\, i, j \,|\, \ell \subseteq D$:



Remark. Monomer-monomer correlations are a special case of dimer models for nontrivial fundamental-group surfaces:



Remark. $|\{[K]\}| = 2^{2g+2n-1}$, where $2n = |$vertices$|$.

1.7 Pfaffian polynomials

Theorem. *Orthonormal sequence exists for 2^{2g} eigenvalues in multiple by*

$$Z = \frac{1}{2^g} \sum_{[K]} \mathsf{Arf}(q^K_{D_0}) \cdot \varepsilon^K(D_0) \cdot \mathsf{Pf}(X^K) \qquad \Big| \, \mathsf{Arf}(\cdot) \in \{\pm 1\}$$

such that:

$[K]$ = *equivalence classes of Kasteleyn orientations*, 2^{2g} *in total*

$q^K_{D_0}$ = *quadratic form on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$; *corresponds to Kasteleyn orientation*, *with respect to reference configuration* D_0

$$\varepsilon^K(D_0) = (-1)^{\sigma}\, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{2n-1}\sigma_{2n}} \quad \left| \begin{array}{l} \sigma \in \mathit{Aut}(\mathscr{D}_0) \subseteq \mathit{Aut}(\mathscr{D}) \\ \{[\sigma]\} \subseteq \mathcal{S}_{2n} \big/ (\mathcal{S}_n \times \mathcal{S}_2^n). \end{array} \right.$$

Proof. $\heartsuit$.

Corollary.

(i). For bipartite graphs on $\overline{\mathcal{M}}_g$:

height function $=$

$=$ *section of the non-trivial* $\mathbb{Z}$*-bundle.*

(ii). Fundamental cycles $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ *is given by*:

$$Z\big(\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_g}, \mathcal{H}_{b_1}, \ldots, \mathcal{H}_{b_g}\big) =$$

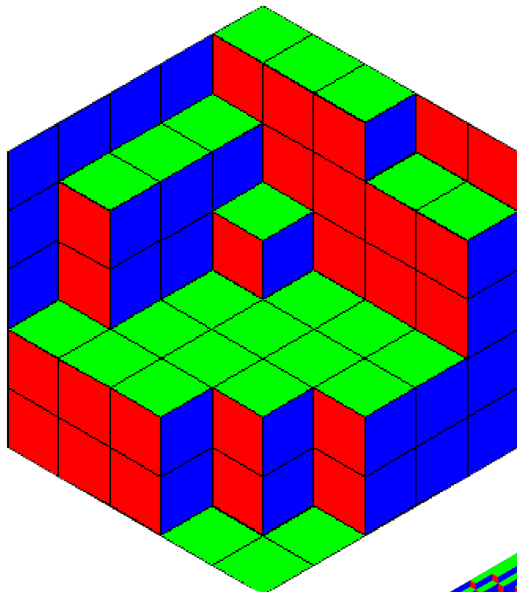
$$= \sum_D \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{i=1}^{g} \exp\Big(\sum_i \mathcal{H}_{a_i} \Delta_{a_i} h + + \sum_i \mathcal{H}_{b_i} \Delta_{b_i} h \Big)$$

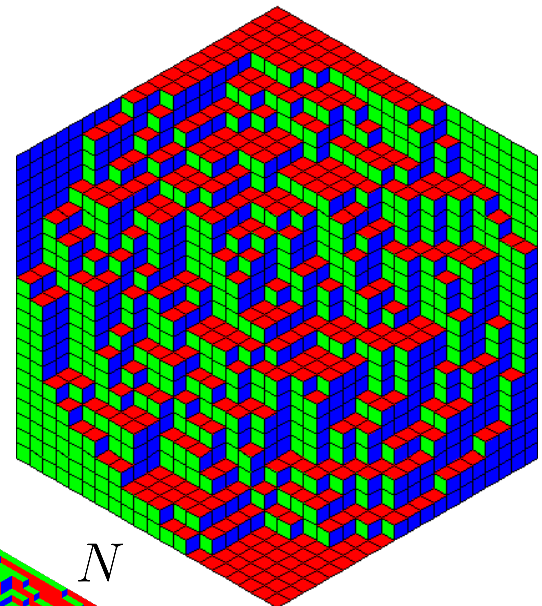
where $\Delta_C h =$

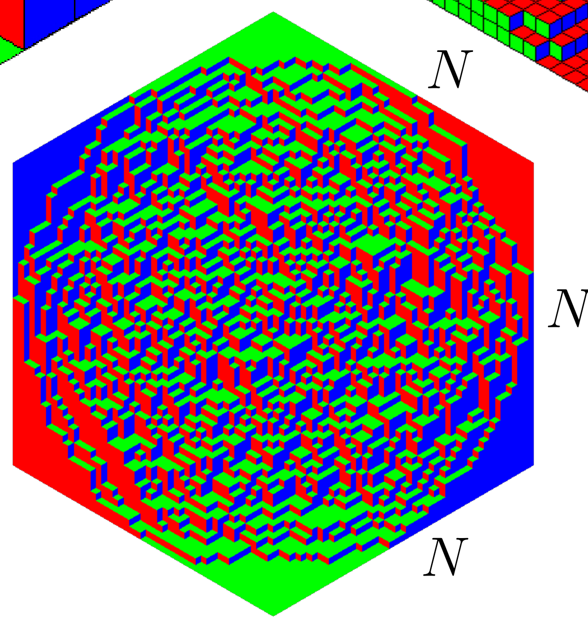
$=$ *change in height function along noncontractible cycle* C *on* $\overline{\mathcal{M}}_g$.

Proof. $\heartsuit$.

1.8 Limits





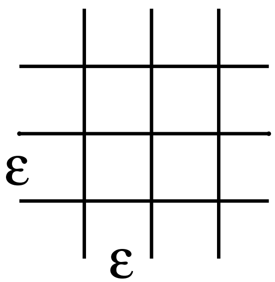


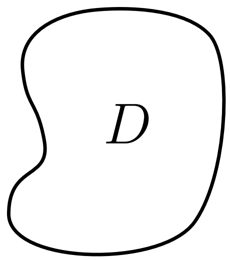
uniform measure

$$\mathrm{Prob}(h) = \frac{1}{|\mathcal{H}_X|}$$

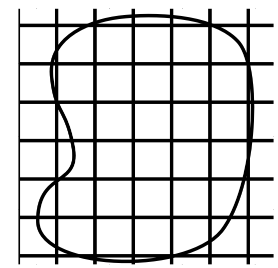
$$N \longrightarrow \infty\,.$$

Theorem (Schur process; Okounkov & R). Let $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \,|\, D \subset \mathbb{R}^2$;





such that: $\begin{cases} \varepsilon \longrightarrow 0, \ |D_\varepsilon| \longrightarrow \infty \\ D_\varepsilon = \varphi_\varepsilon\big(\mathbb{Z}^2\big) \cap D \end{cases}$



.

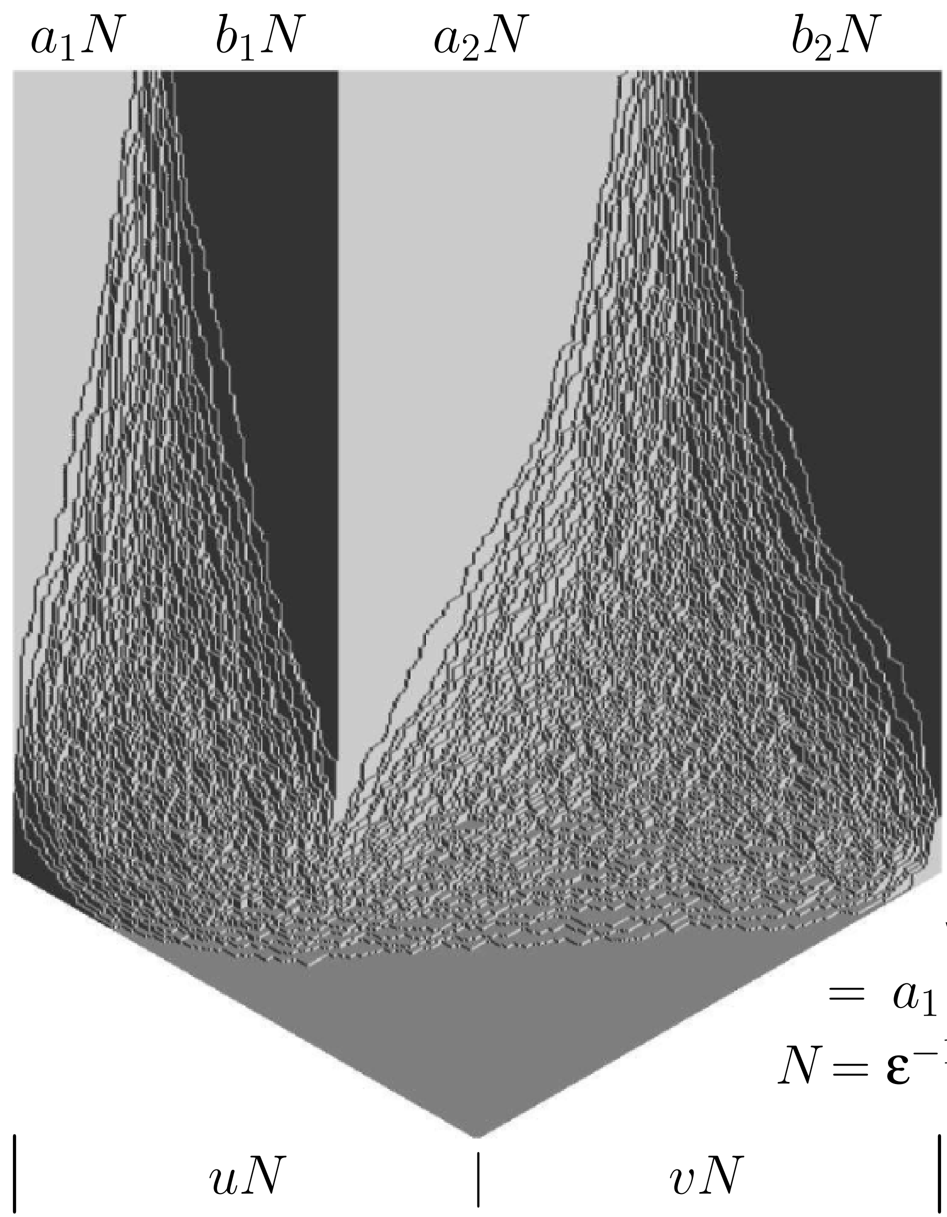
Then, for cube-stack with measure

$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \quad \left| \begin{array}{l} \pi \in \mathcal{H}_X \\ \pi \cong D, \end{array} \right.$$

there is existence of:

$$\textit{Thermodynamic limit } (|D_\varepsilon| \longrightarrow \infty) \ +$$
$$+ \ \textit{Scaling limit } (q = e^{-\varepsilon}, \ \varepsilon \longrightarrow +0).$$

Proof. $\heartsuit$.



where $u+v=$
$= a_1+a_2+b_1+b_2;$
$N=\varepsilon^{-1},\ q=e^{-\varepsilon}.$

2 Special cases

Points:

(i) Formulate the Grassmann kernel in special genus-g domains

(ii) Find thermodynamic $\ln(\cdot)$ scaling, asymptotics variational-principle

(iii) State conjecture for large deviation functional, Green's function $\langle \cdots \rangle$

2.1 Grassmann integral kernels

Pairing, $\bigwedge^{\bullet} V^* \otimes \bigwedge^{\bullet} V \longrightarrow \mathbb{R}$:

$$\langle \varphi(a^*), \psi(a) \rangle \overset{\text{def}}{=\!=} \varphi_0\, \psi_0 + \sum_{k=1}^{2n} \varphi_k \psi_k + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \varphi^{\sigma(k)\cdots\sigma(1)}\, \psi^{\sigma(1)\cdots\sigma(k)} =$$

$$= |\psi_0|^2 + \sum_{k=1}^{2n} \int_{\sigma(k)<} |\psi^{\sigma(1)\cdots\sigma(k)}|^2\, d^{2n}a, \quad \forall\ |\psi|^2 \propto |\varphi|^2 \in \mathbb{R}$$

such that for V basis $(a_1, \ldots, a_{2n})$ and V dual space V^* basis $(a_1^*, \ldots, a_{2n}^*)$:

$$\bigwedge^{\bullet} V \ni \psi(a) = \psi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \psi^{\sigma(k)<}\, a_{\sigma(k)<} \left| \begin{array}{l} \bigwedge^k V \ni \sum \psi^{\sigma(k)<}\, a_{\sigma(k)<} \\ \sigma(k)< \ = \ (\sigma(1) < \cdots < \sigma(k)) \end{array} \right.$$

$$\bigwedge^{\bullet} V^* \ni \varphi(a^*) = \varphi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)>} \varphi^{\sigma(k)>}\, a^*_{\sigma(k)>} \left| \begin{array}{l} \bigwedge^k V^* \ni \sum \varphi^{\sigma(k)<}\, a^*_{\sigma(k)>} \\ \sigma(k)> \ = \ (\sigma(1) > \cdots > \sigma(k)) \end{array} \right.$$

where $\bigwedge^{\bullet} V =$ Grassmann algebra, on basis $(a_1, \ldots, a_{2n}) \subseteq V$, generated by

$$\left\{ \begin{array}{l} a_0 = 1;\ a_{\sigma(k)<} = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},\ \forall\, \sigma(k)< \ = \ (\sigma(1), \ldots, \sigma(k))\ |\ \sigma(1) < \cdots < \sigma(k); \\ a_{\sigma(\xi)} \otimes a_{\sigma(\eta)} + a_{\sigma(\eta)} \otimes a_{\sigma(\xi)} = 0, \forall\, \xi, \eta = 1, \ldots, k;\ \sigma(\xi), \sigma(\eta), k = 1, \ldots, 2n \end{array} \right\}.$$

Fixing integrals on $\bigwedge^\bullet V,\ \bigwedge^\bullet V^*,\ \bigwedge^\bullet(V^*\otimes V)$ by choosing

$$a_1,\dots,a_{2n}\in\bigwedge^{2n}V,\quad a^*_{2n},\dots,a^*_1\in\bigwedge^{2n}V^*$$

and

$$a^*_{2n},\dots,a^*_1,\,a_1,\dots,a_{2n}\in\bigwedge^{2n}V^*\otimes\bigwedge^{2n}V$$

then

$$\int\bigotimes_{i=1}^{\eta}a^*_{\sigma(i)}\ \bigotimes_{i=1}^{\eta}a_{\tau(i)}\ da^*\,da\ =\begin{cases}0 & ,\quad \eta\neq 2n\\ (-1)^{(\sigma\,+\,\tau\,+\,n(2n-1))} & ,\quad \eta=2n\end{cases}$$

$$\sigma\ :\ (\sigma(1),\dots,\sigma(2n))\longrightarrow(1,\dots,2n)$$

$$\tau\ :\ (\tau(1),\dots,\tau(2n))\longrightarrow(1,\dots,2n)\,.$$

Lemma.

$$\left\langle\varphi(a^*),\psi(a)\right\rangle\ =\int\exp\Big(\sum_i a^*_i a_i\Big)\ \varphi(a^*)\ \psi(a)\ da^*\,da\,.$$

Proof. $\heartsuit$.

Lemma. *Let* $A\colon V \longrightarrow V$ *by*

$$\psi_A(a) \;=\; \sum_{\{i\}_<\,,\,\{j\}_<} a_{\{i\}_<}\, A_{\{i\}_<\{j\}_<}\, \psi_{\{j\}_<}$$

$$=\; \psi_0 \oplus A\psi_1 \oplus A^{\otimes 2}\psi_2 \oplus \cdots$$

then

$$\psi_A(b) \;=$$

$$=\; \int \exp(-a^* A\, b)\; \exp(-a^* a)\; \psi(a) da^*\, da.$$

Proof. $\heartsuit$.

Lemma.

$$\int \exp(-a^* A\, b)\; \exp(-a^* a)\; \exp(-B^* B\, a)\; da^* da \;=$$

$$=\; \exp(-b^* B\, A\, b)\,.$$

Proof. $\heartsuit$.

Remark. Therefore, $\exp(-b^* A\, b) =$ "integral kernel" of A acting on $\bigwedge^{2n} V$.

2.2 Vertex operators

(i). The Fermionic Fock space, i.e. $\langle V_m \rangle \in \mathbb{C}^{\,\mathbb{Z}+\frac{1}{2}}$ is given by

$$F = \left\{ V_{m_1} \wedge V_{m_2} \wedge \cdots \;\middle|\; \begin{array}{l} m_i \in \mathbb{Z} + \frac{1}{2} \\ m_{i+1} = m_i - 1 \\ i \gg 1 \end{array} \right\}.$$

(ii). The Clifford algebra:

$$Cl_{\mathbb{Z}} = \left\langle \psi_m \,,\, \psi_m^* \right\rangle \;\left|\; \begin{array}{l} m \in \mathbb{Z} + \frac{1}{2} \\ \psi_m \psi_{m'} + \psi_{m'} \psi_m \;=\; \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* \;=\; 0 \\ \psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m \;=\; \delta_{m\,m'}\,. \end{array} \right.$$

(iii). The Clifford algebra act on the Fock space F:

$$\psi_m \, v_{m_1} \wedge v_{m_2} \wedge \cdots \;=\; v_m \wedge v_{m_1} \wedge v_{m_2} \wedge \cdots$$

$$\psi_m^* \, v_{m_1} \wedge v_{m_2} \wedge \cdots \;=\; \sum_{i=1}^{\infty} (-1)^i \;\delta_{m_i,\,m}\; v_{m_1} \wedge \cdots \wedge \widehat{v_{m_1}} \wedge \cdots$$

(iv). The Heisenberg algebra:

$$\left\langle \alpha_n \right\rangle \quad \left| \begin{array}{l} n \in \mathbb{Z}\backslash\{0\} \\ [\alpha_n,\, \alpha_{n'}] \;=\; -n\,\delta_{n,\,-n'}\,. \end{array} \right.$$

(v). The Heisenberg algebra act on the Fock space F:

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m \,\in\, \mathbb{Z}+\frac{1}{2}} \psi_{m+n}\, \psi^*_m\,.$$

- As operator in F:

$$\Big[\alpha_n\,,\,\psi_\xi\Big] = \psi_{\xi+n} \;\;,\;\; \Big[\alpha_n\,,\,\psi^*_\xi\Big] = -\psi^*_{\xi-n}\,.$$

(vi). The vertex operators in F:

$$X_\pm(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\,\alpha_{\pm n}\Big) \quad \left| \begin{array}{l} \big(X_-(x)v,\, w\big) = \\ \quad = \big(v,\, X_+(x)w\big) = \\ \quad = \big(X_+(x)w,\, v\big). \end{array} \right.$$

(vii). The commutation relations:

$$\begin{aligned} X_+(x)\,X_-(y) &= (1-x) \cdot X_-(y)\,X_+(x) \\ X_+(x)\,\psi(z) &= (1-z^{-1}x)^{-1} \cdot \psi(z)\,X_+(x) \\ X_-(x)\,\psi(z) &= (1-x\,z)^{-1} \cdot \psi(z)\,X_-(x) \\ X_+(x)\,\psi^*(z) &= (1-z^{-1}x) \cdot \psi^*(z)\,X_+(x) \\ X_-(x)\,\psi^*(z) &= (1-z\,x) \cdot \psi^*(z)\,X_-(x). \end{aligned}$$

(viii). The eigenvectors:

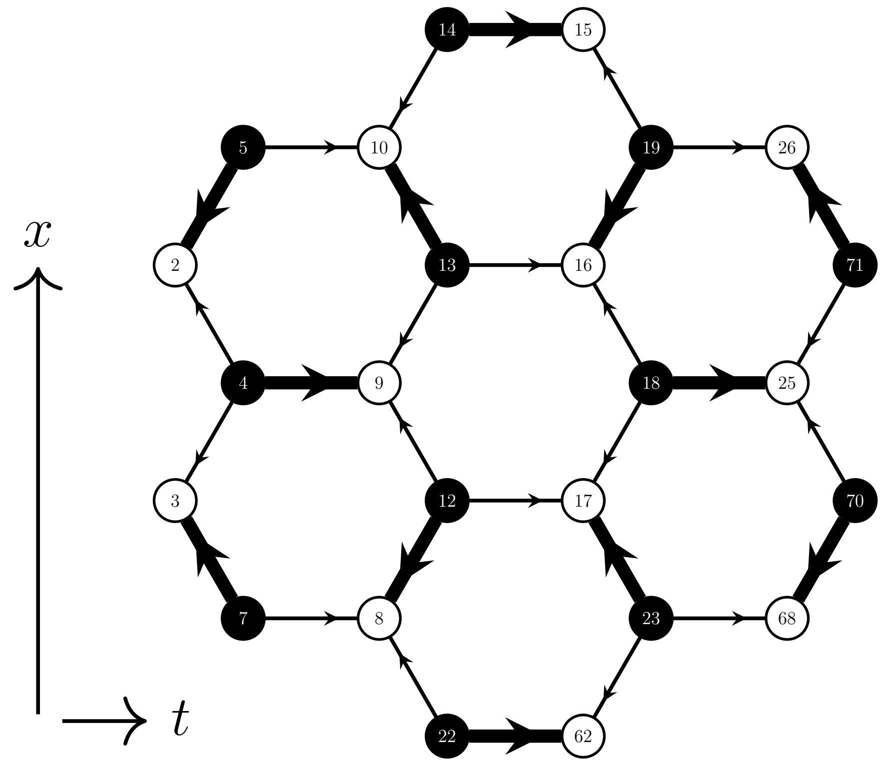
$$X_-(x) \prod_i \psi^*(w_i) \prod_j \psi^*(z_j)\, V_0^{(n)} =$$

$$= \prod_i (1-x\,z_i)^{-1} \prod_j (1-x\,w_j) \prod_i \psi^*(w_i) \prod_j \psi^*(z_j)\, V_0^{(n)}$$

where $V_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

2.3 Fermionic Kasteleyn algebra of operators

By bipartite hexagonal X for the one cube X^* of two-color tiles



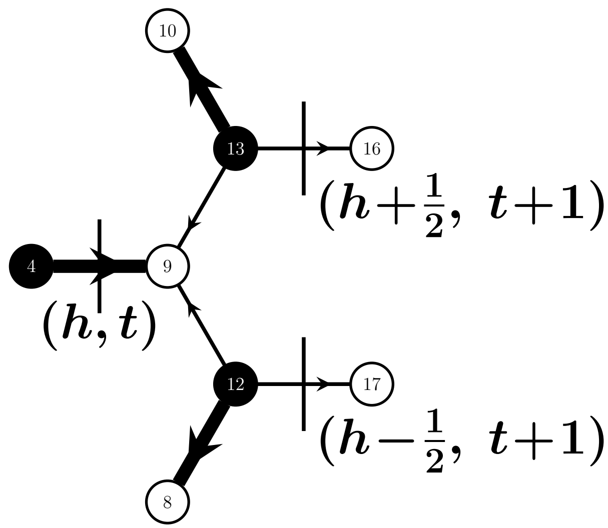
let the general parameterization for bipartite hexagonal lattice be given by:

$$b(h,t) = (h,\ t - \tfrac{1}{2})$$
$$w(h,t) = (h,\ t + \tfrac{1}{2}).$$

Kasteleyn matrix by the above-given $b \sim w$ diagram is then given by

$$K(h,t) \;=\; (h,t) - \big(h+\tfrac{1}{2},\, t+1\big) \;+\; x_{h,t}\,\big(h-\tfrac{1}{2},\, t+1\big).$$

Placing Fermions $a^*_{h,\,t}\,,\; a_{h,\,t}$ respectively at $b(h,t)$ and $w(h,t)$:



$$a^* K a \;=\; \sum_{h,\,t} a^*_{h,t}\, a_{h,t} \;-\; \sum_{h,\,t} a^*_{h+\frac{1}{2},\,t+1}\, a_{h,t} \;+\; \sum_{h,\,t} a^*_{h-\frac{1}{2},\,t+1}\, a_{h,t}\, x_{h,t} \;=$$

$$=\; \sum_{t} \big(a^*_t\, a_t \;+\; a_t\, V\, a^*_{t+1} \;+\; a_t\, V^{-1}\, x_t\; a^*_{t+1}\big).$$

Theorem. *By the algebra of operators, the boundary conditions:*

$$[Diagram] \qquad \left| \begin{array}{l} \textit{Prob}(\pi) \\ \propto \prod_t q_t^{|\pi(t)|} \end{array} \right.$$

assuming $x_{h,t} = x_t,$ *analogous to the notation* $q_{h,t} = q_t,$ *implies*

$$Z = \int \exp\big(a^* A\, a\big)\, da^*\, da \ =$$

$$= \Big\langle X_-\big(x_{-\frac{1}{2}}\big) \cdots X_-\big(x_{u_0+\frac{1}{2}}\big)\, X_+\big(x_{\frac{1}{2}}\big) \cdots X_+\big(x_{u_1+\frac{1}{2}}\big)\, V_0^{(0)}\,, \quad V_0^{(0)} \Big\rangle\ .$$

Proof (outline).

$$\int \cdots \exp\big(a^*_{t-1}\, a_{t-1}\big) \cdot \exp\big(a_{t-1}\,(V - V^{-1} X_t)\, a^*_t\big) \cdot$$
$$\cdot \exp\big(a^*_t\, a_t\big) \cdot \exp\big(a_t\,(V - V^{-1} X_t)\, a^*_{t+1}\big) \cdots \quad =$$

$$= \ \cdots \underbrace{\widetilde{\big(V - V^{-1} X_{t-1}\big)}^{\;-1}}_{X_+(x_t)} \cdot \underbrace{\widetilde{\big(V - V^{-1} X_t\big)}^{\;-1}}_{X_-(x_t)} \cdots$$

where $X_+(x_t)$ and $X_-(x_t)$ each depends on t such that

$$\widetilde{A} = A, \ \text{where } V \hookleftarrow \text{ is lifted to } \textstyle\bigwedge^{\frac{\infty}{2}} V \ \Big|\ V = \bigoplus_{m \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}\, v_h$$

under boundary conditions, etc. □

Remark. Direct proof exists combinatorially besides the Kasteleyn method.

Corollary.

$$Z = \prod_{m=\frac{1}{2}}^{u_1 - \frac{1}{2}} \ \prod_{m' = u_0 + \frac{1}{2}}^{-\frac{1}{2}} \big(1 - x^-_{m'}\, x^+_m\big)^{-1}.$$

Theorem. (Okounkov & R., 2005).

$$\left\langle \boldsymbol{\sigma}_{(h_1 t_1)} \cdots \boldsymbol{\sigma}_{(h_k t_k)} \right\rangle = \det(K((t_i, h_i), (t_j, h_j)))_{1 \leqslant i, j \leqslant k}$$

$$K((t_i, h_i), (t_j, h_j)) =$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{|z| < R(t_1)} \int\limits_{|z| < \widetilde{R}(t_2)} \frac{\Phi_-(z, t_1)\ \ \Phi_+(w, t_2)}{\Phi_+(z, t_1)\ \ \Phi_-(w, t_2)} \cdot$$

$$\cdot \ \frac{1}{z - w} \ \cdot \ z^{\left(-h_1 - B(t_1) - \frac{1}{2}\right)} \cdot \ w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \ dz\, dw$$

where

$|w| < |z|,\ t_1 \geqslant t_2$ | $R(t) = \min\limits_{m > t}((x_m^+)^{-1}),\ \widetilde{R}(t) = \max\limits_{m < t}(x_m^-),\ B(t) = \frac{|t|}{2} - \frac{|t - u_0|}{2}$

$|w| > |z|,\ t_1 < t_2$ | $\Phi_+(z, t) = \prod\limits_{m > \max(t, \frac{1}{2})} (1 - z\, x_m^+),\ \ \Phi_-(z, t) = \prod\limits_{m < \max(t, -\frac{1}{2})} (1 - z^{-1}\, x_m^-).$

Proof. ♡.

2.4 Thermodynamic limit with scaling

$[Diagram]$

$$\left.\begin{aligned} x_m^+ &= aq^m \\ x_m^- &= a^{-1}q^m \end{aligned}\right\} \text{assumed}$$

corresponding to $\text{Prob}(\pi) \propto q^{|\pi|}$.

Considering limit $\varepsilon \longrightarrow 0$, $q = e^{-\varepsilon}$, $u_1 = \varepsilon^{-1} v_1$, $u_0 = \varepsilon^{-1} v_0$ for fixed v_1, v_0:

$$Z = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - x_m^- x_n^+)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - q^{m-n})^{-1}$$

$$\langle\, |\pi|\, \rangle = q \frac{\partial}{\partial q} \ln Z = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1-e^{t-s}}}_{\substack{\text{3D volume} \\ \text{function}}} \, ds\, dt + \cdots$$

where

$$\ln Z = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln(\underbrace{1-e^{-s+t}}_{\substack{\text{2D partition} \\ \text{function}}}) \, ds\, dt + \cdots$$

2.5 Asymptotics of correlation functions

Consider limit $\varepsilon \longrightarrow 0$ where $t_i = \varepsilon^{-1}\tau_i,\ h_1 = \varepsilon^{-1}\chi_i,$ for fixed $\tau_i,\ \chi_i$:

$[Diagram]$ $(\tau_i,\ \chi_i)$ *in the bulk*

$$K((t_1,h_1),(t_2,h_2)) \longrightarrow$$

$$\longrightarrow \frac{1}{(2\pi i)^2} \int_{C_z}\int_{C_w} \exp\big(\varepsilon^{-1}(S(z,t_1,\chi_1) - S(z,t_2,\chi_2))\big) \cdot \cdot\ (zw)^{1/2}\ (z-w)^{-1}\ dz\ dw$$

where

$$S(z,t,\chi) = = -\left(\chi + \frac{\tau}{2} - u_0\right) \ln Z + \mathrm{Li}_2(ze^{-v_0}) + \mathrm{Li}_2(ze^{-v_1}) - \mathrm{Li}_2(z) - \mathrm{Li}_2(ze^{-\tau})$$

and

$$\mathrm{Li}_2(z) = \int_0^z t^{-1} \ln(1-t)\, dt.$$

2.5.1 Critical points

$$\exp\left(\chi + \frac{\tau}{2}\right) = \frac{(1 - ze^{-v_0})(1 - ze^{-v_1})}{(1 - z)(1 - ze^{-\tau})}$$

gives quadratic equation, implying a discriminant for two real solutions or two complex-conjugate solutions, or a zero-discriminant.

$[Diagram]$

$$\boxed{\partial_\chi h_0(\tau, \chi) = \frac{1}{\pi} \mathsf{arg}(z_0)}$$

$$\left\langle \sigma_{(h,t)} \right\rangle = K((t,h),(t,h)) \longrightarrow \varepsilon\, \partial_\chi h_0(\tau, \chi)$$

2.5.2 Steepest descent

$$K((t_1,h_1),(t_2,h_2)) \; = \; -\frac{\varepsilon}{2\pi}\cdot\Big(\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\}}{(z_1-w_2)\sqrt{-w_2S_2''(w_2)}\;\sqrt{z_1S_1''(z_1)}} \; -$$

$$- \; \frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\}}{(z_1-\overline{w}_2)\sqrt{-\overline{w}_2S_2''(\overline{w}_2)}\;\sqrt{z_1S_1''(z_1)}} + c.\,c.\Big)\cdot(1+O(1))$$

That is, for $\mathcal{H}_+=\{z\in\mathbb{C},\ \mathrm{Im}\, z>0\}\mid z_0(\chi,\tau)=$ inner process, such that

$$z_1 \; = z_0(\chi_1,\tau_1)$$

$$w_2 \; = z_0(\chi,\tau),$$

$$K((t_1,h_1),(t_2,h_2)) \; =$$

$$= \; \frac{\varepsilon}{2\pi}\exp\{\varepsilon^{-1}(\mathrm{Re}(S(z_0(\chi_1,\tau_1)))-\mathrm{Re}(S(z_0(\chi_2,\tau_2))))\}\cdot$$

$$\cdot\Big(\frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(w_2)))\}}{(z_1{-}w_2)} \; +$$

$$+ \; \frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(\overline{w}_2)))\}}{(z_1{-}\overline{w}_2)} \; + \; c.\,c.\Big)\cdot(1+O(1)) \quad (*).$$

Hence, solution for Kasteleyn-Fermions to free Dirac-Fermions convergence:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + \\ + \psi_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+O(1))$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^*_{\vec{x}} = \exp(\varepsilon^{-1}\mathsf{Re}(S(z_0)))\cdot\Big(\psi^*_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + \\ + \psi^*_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+O(1))$$

such that

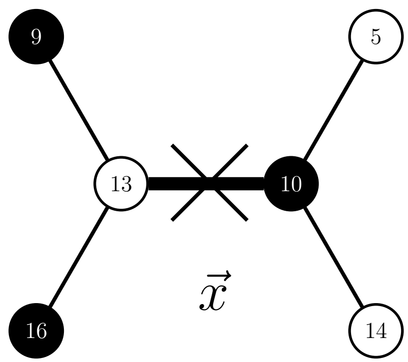
$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\pm(w)) = \frac{1}{z-w}$$

$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\mp(w)) = \mathbb{E}(\psi^*\,\psi^*) = \mathbb{E}(\psi\,\psi) = 0$$

where $\psi^*_\pm(z)$, $\psi_\pm(w)$ are spinors:

$$\psi^*_\pm(z) = \psi^*_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\ , \quad \psi_\pm(z) = \psi_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\ .$$

Remark. The correlation is given by:



$$\Big\langle (\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle)\ (\sigma_{\vec{x}_2} - \langle \sigma_{\vec{x}_2} \rangle) \Big\rangle \ = \ K_{12} K_{21} \ =$$

$$= \frac{\varepsilon^2}{(2\pi)^2} \left(\frac{\dfrac{\partial z_1}{\partial x_1} \dfrac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} \ - \ \frac{\dfrac{\partial z_1}{\partial x_1} \dfrac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} \ + \ c.\,c. \right) \times$$

$$\times \ (1 + O(1)).$$

In particular,

$$\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle = \ \varepsilon\, \partial_x\, \varphi(z_0(\tau, x)) \ + \ \cdots \quad \Big|\ \varphi(z) = \text{Gaussian free field on } \mathcal{H}_+$$

such that the Green's function of Dirichlet problem on $\mathcal{H}_+$ is given by

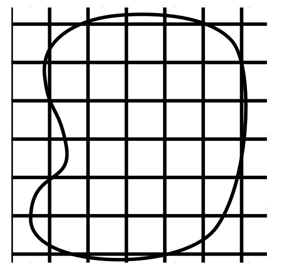
$$\langle \varphi(z)\, \varphi(w) \rangle = \ \frac{1}{2\pi} \ln \left| \frac{z - w}{z - \overline{w}} \right|$$

and, the Bose-Fermi correspondence is given by

$$\partial_x \varphi \ = \ :\widetilde{\psi}(z, \overline{z})\ \ \widetilde{\psi}(z, \overline{z}): \ \cdots \ .$$

2.5.3 Scaling limit by Kasteleyn operator

Let $X = D_\varepsilon = \varphi_\varepsilon(L) \cap D$, for arbitrary lattice $L \mid A_X^K =$ difference operator,



where $\varepsilon \longrightarrow 0$ in the asymptotics of the equation for $G_{x,y}$ given by

$$(A_X^K)_x \cdot G_{x,y} = \delta_{x,y}$$

Cases.

(i) Hexagonal lattice: Utilizes the weighted as above, for

$$q_t = e^{-\varepsilon f(t)}, \quad t = \frac{\tau}{\varepsilon}, \quad \varepsilon \longrightarrow 0.$$

Theorem. $G_{x,y} =$ *same as* $(*)$, *with different* $z_0(\tau, x)$.

Proof. $\heartsuit$.

(ii) Periodic lattice: Utilizes variational principle.

2.6 Variational principle

(i). For the $N \times M$ torus

$[Diagram]$

$$Z(H,V) = \sum_{D} \prod_{\ell} \omega(\ell) \exp(H\Delta_a h_D + V\Delta_b h_D)$$

$$= \frac{1}{2}\Big\{\mathrm{Pf}\big(A^{K_1}\big) + \mathrm{Pf}\big(A^{K_2}\big) + \mathrm{Pf}\big(A^{K_3}\big) - \mathrm{Pf}\big(A^{K_4}\big)\Big\}$$

where $N, M \longrightarrow \infty$, for fixed $\frac{N}{M}$.

And, $\omega(\ell) = 1 \Longrightarrow$ eigenvalues of Kasteleyn matrices by Fourier transform.

Theorem. (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$\lim_{N,M\longrightarrow\infty} \frac{1}{NM} \ln Z_{NM} = \oint\oint \ln|1+zw| \frac{dz}{z}\frac{dw}{w}$$

$$= f(H,V) = \begin{cases} |z| = e^{H} \\ |w| = e^{V}. \end{cases}$$

(ii). Taking Legendre transform

$$\boldsymbol{\sigma}(s,\, t) \;=\; \max_{H,V}\,(H_s + V_t - f(H,\, V))$$

then

$$\sum_D 1 \;=\; \sum_D \prod_D w\,(e) \;=\; \exp(MN\,\boldsymbol{\sigma}(s,\, t)\cdot(1+O(1)))$$

where

$$\frac{\Delta_a h_D}{M} = s, \quad \frac{\Delta_b h_D}{N} = t, \quad M, N \to \infty, \quad \frac{N}{M} \text{ fixed.}$$

(iii). For domain

$$[Diagram]$$

$\Delta_a h = sM,\ \Delta_b h = tN.$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\sum_D 1 \;=\; \exp(MN\,\boldsymbol{\sigma}(s,\, t)\cdot(1+O(1)))$$

with the boundary conditions of height function h_D.

(iv). For domain

$$[Diagram] \qquad M_i \times N_j$$

$$Z_{D_{\boldsymbol{\varepsilon}}} = \sum_{\left\{\substack{\text{values of}\\ \text{height functions}\\ \text{on}\\ \text{boundaries}\\ \text{between rectangles}}\right\}} Z_{\underset{M_i}{\square} N_j} (h_{\text{bound}})$$

$$= \sum_{\{\Delta_x h,\, \Delta_y h\}_{ij}} \exp\left(\sum_{\underset{M_i}{\square} N_j} M_i\, M_j\ \boldsymbol{\sigma}\Big(\frac{\Delta_x h}{M_i}\,,\ \frac{\Delta_y h}{N_j}\Big) \right)$$

$$= \exp\left(\boldsymbol{\varepsilon}^{-2} \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\ dx\, dy\ (1 + O(1))\right)$$

where $h_0 =$ minimizer for

$$S[h] = \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\ dx\, dy.$$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln Z_{D_\varepsilon} = \int_D \sigma(\vec{\nabla} h_0) \, dx \, dy$$

such that

- $h_0 =$ *minimizer*
- $0 < \partial_x h, \partial_y h < 1$
- $h_0\big|_{\partial D} = b,$ *the boundary condition appearing in the limit* $\varepsilon \longrightarrow 0.$

$$[Diagram]$$

for height function

$$\begin{aligned} h &= \varepsilon^{-1} h_0 + \varphi \\ &= \varepsilon^{-1} (h_0 + \varepsilon\varphi) \end{aligned}$$

where $h_0 =$ *limit shape*
and, $\varphi =$ *distribution (factor).*

2.7 Physics way of the distributions

$$S[h_0 + \varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x$$

$$a^{ij}(x) = \partial_i\partial_j\,\varphi(s,t)\begin{cases} s = \partial_1 h_0 \\ t = \partial_2 h_0 \end{cases}$$

such that:

- Partition function equals

$$Z = \exp(\varepsilon^{-2}S(h_0))\int \exp\left(\frac{1}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x\right)D\varphi$$

where $D =$ scalar field with Riemannian metric induced by h_0;

- Correlation equals

$$\langle\varphi(x)\,\varphi(y)\rangle = G(x,y)$$

where $G =$ Green's function for $\Delta = \partial_i(a^{ij}\partial_j)$.

Conjecture. $G =$ *same as obtained by asymptotics of Kasteleyn operators.*

Remark. The conjecture $=$ theorem in certain cases.

Conclusion: Continuum limit phenomena, ongoing

1. How to make such pictures of (i.e. simulate) random configuration:
   (i). Monte Carlo for $\exp(\propto 1000^2)$
   (ii). Sampling around most probable region by MCMC
2. How to describe the continuum limits and distributions analytically:
   (i). Kasteleyn partition and correlation under boundary conditions
   (ii). Variational principle: Minimizing large deviation functional

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Thank you!